The following notes are based on those of Tom Marley's lecture notes from a course on local cohomology in the summer 1999.

1. Refresher on Injective Modules

Recall the following proposition from 902:

Proposition 1.1. *If E is an injective $R-$modules and S is an $R-$algebra, then $\operatorname{Hom}_R(S,E)$ is an injective $S-$module.*

In particular, the proposition shows for an ideal I of R and an injective $R-$module E that $(0:_E I) \cong \operatorname{Hom}_R(R/I, E)$ is an injective $R/I-$module.

Proposition 1.2. *If M is torsion-free and divisible then M is injective.*

Proof. Consider the maps

$$\begin{tikzcd} & & M \\ 0 \arrow[r] & I \arrow[r] \arrow[ur, phantom, ""] \arrow[u, "\phi"] & R \end{tikzcd}$$

Let $i \in I \setminus \{0\}$. Since M is divisible, there exists $x \in M$ such that $\phi(i) = ix$. Let $i' \in I \setminus \{0\}$. Then $\phi(ii') = i\phi(i') = i'\phi(i) = i'ix$. As M is torsion-free, $\phi(i') = i'x$. Define $\tilde{\phi} : R \to M$ by $\tilde{\phi}(r) = rx$. □

Corollary 1.3. *If R is a domain then $Q(R)$ is an injective $R-$module.*

2. Definition of Local Cohomology

Definition. *Let R be a ring, I an ideal, and M an $R-$module. Define*

$$\Gamma_I(M) := \cup_{n\geq 1}^{\infty}(0:_M I^n) = \{m \in M | I^n m = 0 \text{ for some } n\}.$$

Let $f : M \to N$ be an $R-$linear map. Note that $f(\Gamma_I(M)) \subseteq \Gamma_I(N)$ as for $x \in \Gamma_I(M)$ there exists n such that $I^n x = 0$ and so $I^n f(x) = f(I^n x) = 0$. Thus we may define $\Gamma_I(f) = f|_{\Gamma_I(M)} : \Gamma_I(M) \to \Gamma_I(N)$, making $\Gamma_I(-)$ into a covariant functor on the category of $R-$modules.

Proposition 2.1. *$\Gamma_I(-)$ is an additive left exact covariant functor.*

Proof. It is clear that $\Gamma_I(-)$ is additive as the map $\Gamma_I(f)$ is just the restriction map. Thus we are left to prove the left exactness. Suppose $0 \to M \xrightarrow{f} N \xrightarrow{g} L$ is exact and apply $\Gamma_I(-)$: This gives the sequence $0 \to \Gamma_I(M) \xrightarrow{\Gamma_I(f)} \Gamma_I(N) \xrightarrow{\Gamma_I(g)} \Gamma_I(L)$. As $\Gamma_I(f)$ is just the restriction map, we see it is injective. We see $\ker \Gamma_I(g) \supseteq \operatorname{im} \Gamma_I(f)$ as $\Gamma_I(g)\Gamma_I(f) = \Gamma_I(gf) = 0$. Lastly, suppose $x \in \ker \Gamma_I(g) \subseteq \ker g = \operatorname{im} f$. Then there exists $m \in M$ such that $f(m) = x$ and $n \in \mathbb{N}$ such that $I^n x = 0$. So $0 = I^n x = I^n f(m) = f(I^n m)$ implies $I^n m = 0$ as f is injective. Thus $m \in \Gamma_I(M)$ and $x \in \operatorname{im} \Gamma_I(f)$. □

Definition. *The i^{th} **local cohomology** of M with support in I is $H_I^i(M) := R^i\Gamma_I(M)$, where R^iF is the right derived functor of a covariant left exact functor.*

Remarks.

(1) $H_I^i(E) = 0$ if E is injective and $i > 0$.

(2) $H_I^0(E_R(R/p)) = \begin{cases} 0, \text{ if } I \not\subseteq p \\ E_R(R/p), \text{ if } I \subseteq p. \end{cases}$

In particular, this says that since every injective module I is a sum of indecomposable injective modules (that is, $I = \oplus_{p \in \operatorname{Spec} R} E_R(R/p)^{\mu(p,I)}$), we have $H_m^0(I^i) = E_R(R/m)^{\mu_i(M)}$ where $0 \to M \to I^{\cdot}$ is an injective resolution for M.

(3) Every element of $H_I^i(M)$ is killed by a power of I.

Proof. $H_I^i(M) = H_I^0(E^\cdot)$ where $E^\cdot$ is an injective resolution. But every element in $H_I^0(E^i)$ is killed by a power of I. $\square$

(4) Suppose every element of M is killed by a power of I. Then $H_I^0(M) = M$ and $H_I^i(M) = 0$ for $i > 0$.

Proof. Clearly $H_I^0(M) = \Gamma_I(M) = M$. For the latter equality, we first prove the following claim.

Claim. If $\mu_i(p, M) > 0$ then $p \supseteq I$.

Proof. Suppose not. Let $0 \to M \to J^\cdot$ be a minimal injective resolution of M. Then $0 \to M_p \to J_p^\cdot$ is minimal. Since $p \not\supseteq J$, we have $M_p = 0$ and thus $0 \to J_p^\cdot$ is minimal. As each J^i is injective, we see $0 \to J_p^\cdot$ is split exact. Thus

$$\operatorname{Hom}_{R_p}(k(p), J_p^{i-1}) \xrightarrow{0} \operatorname{Hom}_{R_p}(k(p), J_p^i) \xrightarrow{0} \operatorname{Hom}_{R_p}(k(p), J_p^{i+1})$$

is exact and so $\operatorname{Hom}_{R_p}(k(p), J_p^i) = 0$, a contradiction.

Thus $0 \to \Gamma_I(M) \to \Gamma_I(J^\cdot)$ is exact and $H_I^i(M) = 0$ for $i > 0$. $\square$

(5) Let R be Noetherian, M a finitely generated $R-$module. Then $\operatorname{depth}_I M = \min\{i | H_I^i(M) \neq 0\}$.

Proof. Induct on $\operatorname{depth}_I M$. If $\operatorname{depth}_I(M) = 0$, then $I \subseteq Z(M)$ and so $I \subseteq p := (0 : x)$ for $x \neq 0$. So $Ix = 0$ which implies $H_I^0(M) = \Gamma_I(M) \neq 0$. So suppose $t = \operatorname{depth}_I M > 0$. Then I contains a nonzero-divisor on M and so $H_I^0(M) = 0$. Let $x \in I$ be a nonzero-divisor on M. Then we have the exact sequence $0 \to M \xrightarrow{x} M \to M/xM \to 0$. As $\operatorname{depth}_I M/xM = \operatorname{depth}_I M - 1 = t - 1$, inductive gives $J_I^i(M/xM) = 0$ for $i < t-1$ and $0 = H_i^{t-1}(M/xM) \neq 0$. So we have

$$H_I^{i-1}(M/xM) \to H_i^i(M) \xrightarrow{x} H_I^i(M)$$

for $i - 1 < t - 1$. Since $H_I^i(M)$ is killed by some power of x, we have $H_I^i(M) = 0$ for $i < t$. If $i = t$, we have $0 = H_I^{t-1}(M) \to H_I^{t-1}(M/xM) \to H_I^t(M)$ where the middle term is nonzero. Thus $H_I^t(M) \neq 0$. $\square$

Corollary 2.2. *Let (R, m) be local. Then R is Cohen Macaulay if and only if $H_m^i(R) = 0$ for all $i < \dim R$.*

Corollary 2.3. *Let (R, m) be local. Then R is Gorenstein if and only if*

$$H_m^i(R) = \begin{cases} 0, & i \neq \dim R \\ E_R(R/m), & i = \dim R \end{cases}.$$

Proof. Let $I^\cdot$ be a minimal injective resolution of R. By the above remarks, we have $H_m^0(I^i) = E^{\mu_i(R)}$ where $E = E_R(R/m)$.

For the forward direction, suppose R is Gorenstein. Then $\mu_i(R) = 0$ if $i \neq d = \dim R$ and $\mu_d(R) = 1$. So $H_m^0(I^i) = 0$ for $i \neq d$ and $H_m^0(I^d) = E$. Therefore $H_m^d(R) = E$ and $H_m^i(R) = 0$ for all $i \neq d$.

For the backward direction, note R is Cohen Macaulay by the previous corollary. So $\operatorname{Ext}_R^i(R/m, M) = 0$ for all $i < d$, which implies $\mu_i(R) = 0$ for all $i < d$. Thus it is enough to show $\mu_d(R) = 1$. Consider $H_m^0(I^\cdot)$: $0 \to E^{\mu_d(R)} \to E^{\mu_{d+1}(R)} \to \cdots$. By assumption,

$$\begin{tikzcd}
0 \arrow[r] & H_m^d(R) \arrow[r] & E^{\mu_d(R)} \arrow[rr] \arrow[dr, "\phi"] & & E^{\mu_{d+1}(R)} \arrow[r] & \cdots \\
& & & C \arrow[ur] \arrow[dr] & & \\
& & 0 \arrow[ur] & & 0 &
\end{tikzcd}$$

is exact. As $H_m^d(R) \cong E$, we have $E^{\mu_d(R)} \cong H_m^d(R) \oplus C$. Thus $C \cong E^{\mu_d(R)-1}$. Hence $\mu_d(R) = 1$ if and only if $C = 0$.

Apply $\operatorname{Hom}_R(R/m, -)$:

$$
\begin{tikzcd}
\operatorname{Hom}_R(R/m, E^{\mu_d(R)}) \arrow[rr] \arrow[dr, "\tilde{\phi}"'] & & \operatorname{Hom}_R(R/m, E^{\mu_{d+1}(R)}) \arrow[r] & \cdots \\
& \operatorname{Hom}_R(R/m, C) \arrow[ur] \arrow[dr] & & \\
0 \arrow[ur] & & 0 &
\end{tikzcd}
\tag{1}
$$

Note that $\tilde{\phi}$ is surjective as the map ϕ splits.

In general, note that $(0 :_N m) = \operatorname{Hom}_R(R/m, N) \cong \operatorname{Hom}_R(R/m, H^0_m(N)) = (0 :_{H^0_m(N)} m)$ naturally. Hence we have the following commutative diagram

$$
\begin{tikzcd}
\operatorname{Hom}_R(R/m, E^{\mu_d(R)}) \arrow[r] \arrow[d, "\cong"] & \operatorname{Hom}_R(R/m, E^{\mu_{d+1}(R)}) \arrow[d, "\cong"] \\
\operatorname{Hom}_R(R/m, H^0_m(I^d)) \arrow[r] \arrow[d, "\cong"] & \operatorname{Hom}_R(R/m, H^0_m(I^{d+1})) \arrow[d, "\cong"] \\
\operatorname{Hom}_R(R/m, I^d) \arrow[r] & \operatorname{Hom}_R(R/m, I^{d+1})
\end{tikzcd}
$$

where the last map is zero as $I^{\cdot}$ is minimal. Thus, by diagram (1) we see $\operatorname{Hom}_R(R/m, C) = 0$. But $\operatorname{Hom}_R(R/m, C) = \operatorname{Hom}_R(R/m, E^{\mu_d(R)-1}) = K^{\mu_d(R)-1}$. Therefore $\mu_d(R) = 1$ and R is Gorenstein. $\square$

Proposition 2.4. *Let R be Noetherian. Then for any ideal I of R we have $\Gamma_I = \Gamma_{\sqrt{I}}$. In particular, $H^i_I(M) = H^i_{\sqrt{I}}(M)$ for all $i \geq 0$ and for all $R-$modules M.*

Proof. As R is Noetherian, $\sqrt{I}$ is finitely generated. Thus there exists n such that $(\sqrt{I})^n \subseteq I$. Let $x \in \Gamma_{\sqrt{I}}(M)$. Then there exists k such that $(\sqrt{I})^k x = 0$, which implies $I^k x \subseteq (\sqrt{I})^k x = 0$. Therefore $x \in \Gamma_I(M)$.

Let $x \in \Gamma_I(M)$. Then there exists k such that $I^k x = 0$. Since $(\sqrt{I})^n \subseteq I$, $(\sqrt{I})^{kn} \subseteq I^k$ and so $(\sqrt{I})^{kn} x = 0$. Therefore $x \in \Gamma_{\sqrt{I}}(M)$. $\square$

Proposition 2.5. *Let R be Noetherian, S a multiplicatively closed set, M an $R-$module, and I an ideal. Then $H^i_I(M)_S \cong H^i_{I_S}(M_S)$ for all i.*

Proof. Recall that $H^i_I(M)_S$ is computed by taking an injective resolution of M, applying $H^0_I(-)$, taking homology and then localizing. As localization is flat, it commutes with taking homology. Thus it is enough to show localization commutes with the functor $H^0_I(-)$, that is, $H^0_I(M)_X = H^0_{I_S}(M_S)$. Clearly $H^0_I(M)_S \subseteq H^0_{I_S}(M_S)$. Suppose $(I_S)^n \cdot (\frac{m}{s}) = 0$. As I is finitely generated there exists $s' \in S$ such that $s' I^n m = 0$ This implies $s'm \in H^0_I(M)$ and so $\frac{m}{s} \in H^0_I(M)_S$. $\square$

Proposition 2.6. *Let (R, m) be a local ring, M a finitely generated $R-$module. Then $H^i_m(M)$ is Artinian for all i.*

Proof. Let $0 \to M \to I^{\cdot}$ be a minimal injective resolution of M. As $H^0_m(I^i) = E_R(R/m)^{\mu_i(M)}$, $\mu_i(M) < \infty$, and $E_R(R/m)$ is Artinian, we see $H^0_m(I^i)$ is Artinian and $H^i_m(M)$ is a subquotient of $H^0_m(I^i)$. $\square$

Proposition 2.7. *Let I be an ideal, M an $R-$module. Then $H^i_I(M) \cong \varinjlim \operatorname{Ext}^i_R(R/I^n, M)$.*

Proof. For $i = 0$, note that $\operatorname{Hom}_R(R/I^n, M) \cong (0 :_M I^n)$. $\square$

3. "A Note on Factorial Rings" Murthy, 1964

The goal of this section is to prove the following theorem, but to do so we must first prove a series of lemmas.

Theorem 3.1. *Let A be a UFD which is a quotient of a regular local ring. Then TFAE*

(1) *A is Cohen Macaulay*
(2) *A is Gorenstein*

From now on, let B be a regular local ring, $n = \dim B$, $A = B/p$ where $p \in \operatorname{Spec} B$ and $r = \operatorname{ht} p$.

Lemma 3.2. *Let M be a Cohen Macaulay $B-$module and $h = \operatorname{pd}_B M$. Then $\operatorname{Ext}^i_B(M, B) = 0$ for all $i < h$ and $M' = \operatorname{Ext}^h_B(M, B)$ is Cohen Macaulay with $\operatorname{pd}_B M = h$.*

Proof. See Proposition 3.3.3 in BH, or my reading course notes. $\square$

Lemma 3.3. *Let M be a finitely generated $B-$module. Then $p \in \operatorname{Ass} M$ implies $\operatorname{pd}_B M \geq \operatorname{ht} p$.*

Proof. Since B is a regular local ring, $\operatorname{pd}_B M = \dim B - \operatorname{depth}_B M$ and $\operatorname{ht} p = \dim B - \dim B/p$. Thus $\operatorname{pd}_B M \geq \operatorname{ht} p$ if and only if $\operatorname{depth}_B M \leq \dim B/p$. But if $p \in \operatorname{Ass} M$, this inequality holds. $\square$

Lemma 3.4. *Suppose $A = B/p$ is a Cohen Macaulay ring. Then $M := \operatorname{Ext}^r_B(A, B) \cong A$ or an unmixed height one ideal.*

Proof. Recall an ideal I is unmixed if every member of $\operatorname{Ass}_B B/I$ has the same height. We will prove by induction on $\ell = \dim A = \dim B/p = n - r$. First suppose $\ell = 0$. Then $p = m_B$ and so $M = \operatorname{Ext}^n_B(B/m, B) \cong B/m = A$. Now suppose $\ell > 0$. Then $p \neq m_B$. Let $\overline{q} = q/p \in \operatorname{Spec} A$ where $p \subsetneq q \subsetneq m_B$. We have $M_{\overline{q}} = \operatorname{Ext}^r_{B_q}(A_{\overline{q}}, B_{\overline{q}})$. By induction, $M_{\overline{q}}$ is a torsion-free $A_{\overline{q}}-$module of rank 1. Thus $\overline{q} \not\subseteq \operatorname{Ass}_A M$. So $\operatorname{Ass}_A M \subseteq \{(0), \overline{m}\}$. Since A is Cohen Macaulay, $\operatorname{depth} A = \dim A = \ell$. Then $\operatorname{pd}_B A = \dim B - \operatorname{depth} A = \dim B - \dim A = n - \ell < \dim B$. By the lemma above, $M = \operatorname{Ext}^r_B(A, B)$ is Cohen Macaulay and $\operatorname{pd}_B M = r$. Hence $\operatorname{depth}_A M = \operatorname{depth}_B M = \dim B - \operatorname{pd}_B M = \dim B - r > 0$. Therefore $\overline{m} \not\subseteq \operatorname{Ass} M$. Hence $\operatorname{Ass} M = \{(0)\}$ and M is torsion free. Now $M_{(0)} = M_p = \operatorname{Ext}^r_B(A, B)_p = \operatorname{Ext}^r_{B_p}(k(p), B_p) = k(p)$. So $\operatorname{rank}_A M = 1$. Thus $M \cong \overline{I}$ where $I \subseteq B$ is an ideal. If $I = B$, then $M \cong A$ and we are done. So suppose I is proper. We have the following short exact sequences:

(a) $0 \to p \to B \to A \to 0$
(b) $0 \to p \to I \to I_p \to 0$ where $I_p = \overline{I} \cong M$
(c) $0 \to I \to B \to B/I \to 0$

From (a), $\operatorname{pd}_B p = \operatorname{pd}_A p - 1 = r - 1$. We already have $\operatorname{pd}_B M = r$ and so from (b) and the Horseshoe Lemma we get $\operatorname{pd}_B I \leq r$. Then by (c) we have $\operatorname{pd}_B B/I \leq r + 1$. By the previous lemma, if $q \in \operatorname{Ass} B/I$ then $\operatorname{ht} q \leq \operatorname{pd} B/I \leq r + 1$. Therefore I is unmixed of height $r + 1$. Hence $M \cong \overline{I} = I/p$ is unmixed of height 1. $\square$

Proof of Theorem 3.1. We need only show that A Cohen Macaulay implies A is Gorenstein. Write $A = B/p$ as in the theorem. By the last lemma, $\operatorname{Ext}^r_B(A, B) \cong A$ or $\overline{I}$ where $\overline{I}$ is an unmixed ideal of height 1. If $\operatorname{Ext}^r_B(A, B) \cong A$, then we are done as $\omega_A = \operatorname{Ext}^r_B(A, \omega_B) = \operatorname{Ext}^r_B(A, B) \cong A$. So suppose $\operatorname{Ext}^r_B(A, B) \cong \overline{I}$. Recall that height 1 primes are principal in a UFD. So $\overline{I}$ is principal which implies $\overline{I} \cong A$. $\square$

4. The Tensor Product of Co-complexes

Let $C^\cdot, D^\cdot$ be two co-complexes. Define $(C \otimes_R D)^\cdot$ by $(C \otimes_R D)^n := \oplus_{i+j=n} C^i \otimes_R D^j$ and define a map ∂ on $C \otimes_R D$ as follows: for $c \otimes d \in C^i \otimes D^j$, let $\partial(c \otimes d) = \partial c \otimes d + (-1)^i c \otimes \partial d$. Note here that $\partial^2 = 0$.

Facts.

(1) $(C \otimes_R D)^\cdot \cong (D \otimes_R C)^\cdot$ as complexes.
(2) $C \otimes (D \otimes E) \cong (C \otimes D) \otimes E$.

Definition. *Let* $\underline{x} = x_1, ..., x_n \in R$. *Define the* ***Čech complex*** *on* R *with respect to* $x_1, ..., x_r$ *by*

$$\begin{array}{lcl} C^{\cdot}(x_1; R) & := & 0 \to R \to R_{x_1} \to 0 \text{ where } r \mapsto \frac{r}{1} \\ C^{\cdot}(x_1, ..., x_n; R) & := & C^{\cdot}(x_1, ..., x_{n-1}; R) \otimes_R C^{\cdot}(x_n; R) \\ & = & \otimes_{i=1}^n C^{\cdot}(x_i; R) \end{array}$$

Example. Lets compute $C^{\cdot}(x, y; R)$: By the above, we get the sequence

$$0 \to R \otimes R \xrightarrow{f} R_x \otimes R \oplus R \otimes R_y \xrightarrow{g} R_x \otimes R_y \to 0$$

where $f(1 \otimes 1) \mapsto \frac{1}{1} \otimes 1 \oplus 1 \otimes \frac{1}{1}$, $g(\frac{1}{1} \otimes 1, 0) = (-1)\frac{1}{1} \otimes \frac{1}{1}$, and $g(0, 1 \otimes \frac{1}{1}) = \frac{1}{1} \otimes \frac{1}{1}$. Simplifying this, we get

$$0 \to R \xrightarrow{f} R_x \oplus R_y \xrightarrow{g} R_{xy} \to 0$$

where $f(1) = (1, 1)$, $g(1, 0) = -1$ and $g(0, 1) = 1$. In general, $C^{\cdot}(\underline{x}; R)$ looks like

$$0 \to \overset{0}{R} \to \oplus_{i=1}^n R_{x_i} \to \oplus_{i<j} R_{x_i x_j} \to \cdots \to R_{x_1 \cdots x_n} \to 0$$

where the differentials are the same as the maps in the Koszul co-complex with 1's in the place of the x_i's.

Definition. *If* M *is an* $R-$*module, we define* $C^{\cdot}(\underline{x}; M) := C^{\cdot}(\underline{x}; R) \otimes_R M$. *The* i^{th} ***Čech cohomology*** *of* M *is* $H^i_{\underline{x}}(M) := H^i(C^{\cdot}(\underline{x}; M))$.

We want to show $H^i_{\underline{x}}(M) = H^i_{(x)}(M)$, that is, the Čech Cohomology and local cohomology for M are the same. We will start by proving the claim for $i = 0$ and later show for $i \geq 0$.

Lemma 4.1. *Let* M *be an* $R-$*module,* $\underline{x} = x_1, .., x_n \in R$, $I = (\underline{x})$. *Then* $H^0_{\underline{x}}(M) \cong H^0_I(M)$.

Proof. From the above, $C^{\cdot}(\underline{x}; M)$ starts out as $0 \to M \xrightarrow{\partial_0} \oplus_{i=1}^n M_{x_i}$. Now

$$\begin{array}{lcl} m \in H^0_{\underline{x}}(M) & \Leftrightarrow & m \in \ker \partial_0 \\ & \Leftrightarrow & \frac{m}{1} = 0 \text{ in } M_{x_i} \text{ for all } i \\ & \Leftrightarrow & \text{there exists } t \geq 0 \text{ such that } x_i^t m = 0 \text{ for all } i \\ & \Leftrightarrow & \text{there exists } t \geq 0 \text{ such that } I^t m = 0 \\ & \Leftrightarrow & m \in H^0_I(M). \end{array}$$

$\square$

Proposition 4.2. *Suppose* $0 \to L \to M \to N \to 0$ *is a short exact sequence of* $R-$*modules and* $\underline{x} = x_1, ..., x_n \in R$. *Then there exists a natural long exact sequence*

$$\cdots \to H^n_{\underline{x}}(L) \to H^n_{\underline{x}}(M) \to H^n_{\underline{x}}(N) \to H^{n+1}_{\underline{x}}(L) \to \cdots$$

Proof. Consider the following commutative diagram with exact rows and columns (the columns are exact as localization is).

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & 0 \arrow[d] & \\
0 \arrow[r] & L \arrow[r] \arrow[d] & \oplus L_{x_i} \arrow[r] \arrow[d] & \cdots \arrow[r] & L_{x_1\cdots x_n} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & M \arrow[r] \arrow[d] & \oplus M_{x_i} \arrow[r] \arrow[d] & \cdots \arrow[r] & M_{x_1\cdots x_n} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & N \arrow[r] \arrow[d] & \oplus N_{x_i} \arrow[r] \arrow[d] & \cdots \arrow[r] & N_{x_1\cdots x_n} \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & & 0 &
\end{tikzcd}$$

This gives us the short exact sequence of co-complexes: $0 \to C^{\cdot}(\underline{x};L) \to C^{\cdot}(\underline{x};M) \to C^{\cdot}(\underline{x};N) \to 0$. The long exact sequence now follows. $\square$

Proposition 4.3. *Let M be an $R-$module and $\underline{x} = x_1,...,x_n \in R$. Let $y \in R$. Then there exists a long exact sequence*

$$\cdots H^i_{\underline{x},y}(M) \to H^i_{\underline{x}}(M) \xrightarrow{(-1)^i} H^i_{\underline{x}}(M)_y \to H^{i+1}_{\underline{x},y}(M) \to \cdots.$$

Proof. Let $C^{\cdot} = C^{\cdot}(\underline{x};M)$ and $C^{\cdot}(y) = C^{\cdot}(\underline{x},y;M) = C^{\cdot}(\underline{x};M) \otimes C^{\cdot}(y;R)$. Then $C^{\cdot}(y) = C^{\cdot} \otimes (0 \to \overset{0}{R} \to \overset{1}{R_y} \to 0)$. Hence $C^{\cdot}(y)^n = C^{n-1} \otimes_R Ry \oplus C^n \otimes_R R \cong C^{n-1}_y \oplus_R C^n$. Consider the following commutative diagram.

$$\begin{tikzcd}
0 \arrow[r] & C^{n-1}_y \arrow[r] \arrow[d, "\partial_y"] & C^{n-1}_y \arrow[d, "\partial_y"'] & \oplus\, C^n \arrow[r] \arrow[d, "\partial"] \arrow[dl, "(-1)^n"'] & C^n \arrow[r] \arrow[d, "\partial"] & 0 \\
0 \arrow[r] & C^n_y \arrow[r] & C^n_y & \oplus\, C^{n+1} \arrow[r] & C^{n+1} \arrow[r] & 0
\end{tikzcd}$$

This yields the short exact sequence of co-complexes: $0 \to C^{\cdot}_y[-1] \to C^{\cdot}(y) \to C^{\cdot} \to 0$, which gives the long exact sequence

$$\begin{tikzcd}
\cdots \arrow[r] & H^{i-1}_{\underline{x}}(M)_y \arrow[r] & H^i_{\underline{x},y}(M) \arrow[r] & H^i_{\underline{x}}(M) \arrow[r, "\partial"] & H^i_{\underline{x}}(M)_y \arrow[r] & \cdots \\
 & H^{i-1}(C^{\cdot}_y) \cong H^{i-1}(C^{\cdot}(y)) \arrow[u] & & & &
\end{tikzcd}$$

where ∂ is the connecting homomorphism given by the snake lemma applied to the previous diagram. It is clear that $\partial = (-1)^n$. $\square$

Corollary 4.4. *Let M be an $R-$module and $x_1,..,x_n \in R$. Suppose some x_i acts as a unit on M (that is, M is an $R_{x_i}-$module). Then $H^i_{\underline{x}}(M) = 0$ for all i.*

Proof. For $i = 0$, it is clear that $H^i_{\underline{x}}(M) = H^0_{(\underline{x})}(M) = 0$. So suppose $i > 0$. As $C^{\cdot}(\underline{x};M) = [\otimes^n_{i=1} C^{\cdot}(x_i;R)] \otimes_R M$, we may assume without loss of generality that x_n acts as a unit on M. Let $\underline{x}' = x_1,...,x_{n-1}$. By the proposition, there exists a long exact sequence $\cdots \to H^i_{\underline{x}}(M) \to H^i_{\underline{x}'}(M) \xrightarrow{(-1)^i} H^i_{\underline{x}'}(M)_{x_n} \to \cdots$. As M is an $R_{x_n}-$module, each module in $C^{\cdot}(\underline{x}';M)$ is an $R_{x_n}-$module. Hence the map $H^i_{\underline{x}'}(M) \xrightarrow{(-1)^i} H^i_{\underline{x}'}(M)_{x_n}$ defined by $m \mapsto (-1)^i \frac{m}{1}$ is an isomorphism for all i. Therefore, $H^i_{\underline{x}}(M) = 0$ for all i. $\square$

Proposition 4.5. *Let R be a Noetherian ring, $\underline{x} = x_1,...,x_n \in R$. For any injective $R-$module I, $H^i_{\underline{x}}(I) = 0$ for all $i \geq 1$.*

Proof. As $I = \oplus E_R(R/p)$, it is enough to show the proposition in the case $E = E_R(R/p)$ for some $p \in \operatorname{Spec} R$.

Case 1. $x_1, .., x_n \in p$. As every element in E is annihilated by a power of p, $E_{x_i} = 0$ for all i. Thus $C^{\cdot}(\underline{x}; E) = 0 \to E \to 0 \to 0 \to \cdots$. So $H^0_{\underline{x}}(E) = E$ and $H^i_{\underline{x}}(E) = 0$ for all $i \geq 1$.

Case 2. There exists $x_i \notin p$. Then x_i acts as a unit on E and hence $H^i_{\underline{x}}(E) = 0$ for all $i \geq 1$ by the corollary.

$\square$

Theorem 4.6. *Let R be Noetherian, $I = (x_1, ..., x_n)$, M any $R-$module. Then there exists a natural isomorphism $H^i_{\underline{x}}(M) \cong H^i_I(M)$ for all $i \geq 0$.*

Proof. We will induct on i. We have already shown the claim for $i = 0$. So suppose $i > 0$. Let $E = E_R(M)$ and consider the short exact sequence $0 \to M \to E \to C \to 0$. Then there exists a long exact sequence

$$\begin{tikzcd}
\cdots \arrow[r] & H^{i-1}_{\underline{x}}(E) \arrow[r] \arrow[d, "\cong"] & H^{i-1}_{\underline{x}}(C) \arrow[r] \arrow[d, "\cong"] & H^u_{\underline{x}}(M) \arrow[r] \arrow[d, dashed] & H^i_{\underline{x}}(E) = 0 \\
\cdots \arrow[r] & H^{i-1}_I(E) \arrow[r] & H^{i-1}_I(C) \arrow[r] & H^i_I(M) \arrow[r] & H^i_I(E) = 0
\end{tikzcd}$$

By the Five Lemma, $H^i_{\underline{x}}(M) \cong H^i_I(M)$. $\square$

5. Local Cohomology and Arithmetic Rank

Definition. *If I is an ideal of R, the **arithmetic rank** of I, denoted* $\operatorname{ara}(I)$, *is defined by*

$$\operatorname{ara}(I) = \min\{n \geq 0 | \text{ there exists } a_1, ..., a_n \text{ such that } \sqrt{I} = \sqrt{(a_1, ..., a_n)}\}.$$

Corollary 5.1. *Let I be an ideal of a Noetherian ring R and M an $R-$module. Then $H^i_I(M) = 0$ for all $i >$* $\operatorname{ara}(I)$.

Proof. Let $t = \operatorname{ara}(I)$. Then there exists $a_1, ..., a_t \in R$ such that $\sqrt{(a_1, ..., a_t)} = \sqrt{I}$. Then

$$H^i_I(M) \cong H^i_{\sqrt{I}}(M) \cong H^i_{\sqrt{(\underline{a})}}(M) \cong H^i_{(\underline{a})}(M) = H^i_{\underline{a}}(M) = 0$$

for $i > t$. $\square$

Definition. *Let R be a Cohen Macaulay local ring and p a prime of height h. Then p is called a **set theoretic complete intersection** if* $\operatorname{ara}(p) = h$.

Corollary 5.2. *Let R be Cohen Macaulay,* $\operatorname{ht} p = h$ *and $H^{h+1}_p(R) \neq 0$. Then p is* not *a s.t.c.i.*

Example. Let $R = k[x_{ij}]_{1 \leq i \leq 2, 1 \leq j \leq 3}$ with *char* $k = 0$. Let $I = I_2((x_{ij}))$, the ideal of 2×2 minors of the matrix (x_{ij}). Then I is prime of height 2. Hochster proved that $H^3_I(R) \neq 0$ and so I is *not* a s.t.c.i.

Lemma 5.3. *Let R be a Noetherian ring, I an ideal. For any integer $r \geq 1$, there exists $f_1, ..., f_r \in I$ such that for any prime p with* $\operatorname{ht} p \leq r - 1$ *we have $p \supseteq I$ if and only if $p \supseteq (f_1, ..., f_r)$.*

Proof. We will induct on r. If $r = 1$, choose $f_1 \in I \setminus \cup P_i$ where the union ranges over all primes with $\operatorname{ht} p_i = 0$ and $I \not\subseteq P_i$. Now suppose $r > 1$. By induction, we have $f_1, ..., f_{r-1} \in I$ such that if $\operatorname{ht} p \leq r - 2$ then $p \supseteq (f_1, ..., f_{r-1})$ if and only if $p \supseteq I$. Choose $f_r \in I \setminus \cup p_i$ where now the union ranges over all primes p_i minimal over $(f_1, .., f_{r-1})$ with $\operatorname{ht} p_i = r - 1$ and $I \not\subseteq p_i$.

Claim. $(f_1, ..., f_r)$ works.

Proof. Let $p \supseteq (f_1, ..., f_r)$ with $\operatorname{ht} p \leq r - 1$. If $\operatorname{ht} p \leq r - 2$, then we are done by induction. So suppose $\operatorname{ht} p = r - 1$. If p is not minimal over $(f_1, ..., f_{r-1})$, then there exists a prime q with $p \supsetneq q \supseteq (f_1, ..., f_{r-1})$. So $\operatorname{ht} q \leq r - 2$ and $q \supseteq I$. If p is minimal over $(f_1, ..., f_{r-1})$ then $I \subseteq p$ by choice of f_r.

$\square$

Theorem 5.4. *Let R be a Noetherian ring of dimension d and I an ideal of R. Then $\operatorname{ara}(I) \leq d+1$. If R is local, then $\operatorname{ara}(I) \leq d$.*

Proof. By the lemma, there exists $f_1, ..., f_{d+1} \in I$ such that for all $p \in \operatorname{Spec} R$, $p \supseteq I$ if and only if $p \supseteq (f_1, ..., f_{d+1})$. Hence $\sqrt{I} = \sqrt{(f_1, ..., f_{d+1})}$. If (R, m) is local, we know there exists $f_1, ..., f_d \in I$ such that for all $p \neq m$ we have $p \supset (f_1, ..., f_d)$ if and only if $p \supset I$. Since m contains both ideals, $\sqrt{I} = \sqrt{(f_1, ..., f_d)}$. $\square$

Theorem 5.5. *Let R be a Noetherian ring of dimension d, I an ideal, and M an $R-$module. Then $H^i_I(M) = 0$ for all $i > d$.*

Proof. If R is local, then $\operatorname{ara}(I) \leq d$. Otherwise, let $p \in \operatorname{Spec} R$. Then for $i > d$ we have $H^i_I(M)_p \cong H^i_{IR_p}(M_p) = 0$ as $\dim R_p \leq d$. Hence $H^i_I(M) = 0$ for all $i > d$. $\square$

Theorem 5.6 (Change of Rings Principle). *Let S be an $R-$algebra, where R and S are Noetherian. Let I be an ideal of R and M an $S-$module. Then $H^i_I(M) \cong H^i_{IS}(M)$ for all i where we consider M as an $R-$module on the left hand side and as an $S-$module on the right hand side.*

Proof. Let $I = (x_1, ..., x_n)R$. Then, consider the Čech complex, we have

$$C^{\cdot}_R(\underline{x}; M) = C^{\cdot}(\underline{x}; R) \otimes M = C^{\cdot}(\underline{x}; R) \otimes_R (S \otimes_S M) = C^{\cdot}(\underline{x}; S) \otimes_S M = C^{\cdot}_S(\underline{x}; M).$$

Thus $H^i_I(M) = H^i_{\underline{x}}(M) = H^i_{\underline{x}S}(M) = H^i_{IS}(M)$. $\square$

Corollary 5.7. *Let R be a Noetherian ring, I an ideal of R and M a finite $R-$module. Then $H^i_I(M) = 0$ for all $i > \dim M$.*

Proof. Recall $\dim M = \dim R/\operatorname{Ann}_R M$ and M is an $R/\operatorname{Ann}_R M-$module. Thus $H^i_I(M) \cong H^i_{IS}(M)$ where $S = R/\operatorname{Ann}_R M$. Hence $H^i_{IS}(M) = 0$ for $i > \dim S$. $\square$

Proposition 5.8. *Let S be a flat $R-$algebra with R, S Noetherian. Let I be an ideal of R and M an $R-$module. Then $H^i_I(M) \otimes_R S \cong H^i_{IS}(M \otimes_R S)$ for all $i \geq 0$.*

Proof. We have

$$\begin{aligned} H^i_I(M) \otimes_R S &= H^i(C^{\cdot}(\underline{x}; M)) \otimes_R S \text{ where } I = (\underline{x})R \\ &\cong H^i(C^{\cdot}(\underline{x}; M) \otimes_R S) \text{ (since } S \text{ is flat, } - \otimes_R S \text{ is exact)} \\ &\cong H^i(C^{\cdot}(\underline{x}S; M \otimes_R S)) \\ &= H^i_{\underline{x}S}(M \otimes_R S) \\ &= H^i_{IS}(M \otimes_R S). \end{aligned}$$

$\square$

Corollary 5.9. *Let (R, m) local, I an ideal, M a finite $R-$module. Let $\hat{R}$ be the $m-$adic completion of R. Then $H^i_I(M) \otimes_R \hat{R} \cong H^i_{I\hat{R}}(M \otimes_R \hat{R}) \cong H^i_{I\hat{R}}(\hat{M})$ for all i.*

Proposition 5.10. *Let R be Noetherian, M be an $R-$module, and $I = (x_1, ..., x_n)$ an ideal. Then $H^n_I(M) \cong M_{x_1 \cdots x_n} / \sum_{i=1}^n M_{x_1 \cdots \hat{x_i} \cdots x_n}$.*

Proof. Recall that $H^n_I(M)$ is the homology of $\oplus_i M_{x_1 \cdots \hat{x_i} \cdots x_n} \xrightarrow{\phi} M_{x_1 \cdots x_n} \to 0$ where $\phi(0, ..., w, ..., 0) = (-1)^i w$. Therefore, $\operatorname{im} \phi = \sum_i M_{x_1 \cdots \hat{x_i} \cdots x_n} \subseteq M_{x_1 \cdots x_n}$. Hence $H^n_I(M) = M_{x_1 \cdots x_n} / \sum M_{x_1 \cdots \hat{x_i} \cdots x_n}$. $\square$

Corollary 5.11. *Let (R, m) be a Gorenstein local ring and $x_1, ..., x_d$ be a system of parameters for R. Then $E_R(R/m) \cong R_{x_1 \cdots x_d} / \sum_i R_{x_1 \cdots \hat{x_i} \cdots x_d}$.*

Proof. $H^d_{(\underline{x})}(R) = H^d_m(R) \cong E_R(R/m)$. $\square$

Example. Let $R = k[x_1, ..., x_d]$ for a field k and $m = (x_1, ..., x_d)$. Then $E_R(R/m) = R_{x_1 \cdots x_d} / \sum R_{x_1 \cdots \hat{x_i} \cdots x_d} \cong \oplus_{i_1, ..., i_d \in \mathbb{N}^d} k x_1^{-i_1} \cdots x_d^{-i_d}$.

6. Direct Limits and Koszul Cohomology

Theorem 6.1. *Let $I \subseteq R$, M an $R-$module. Then $H^i_I(M) \cong \varinjlim \operatorname{Ext}^i_R(R/I^n, M)$ for all i.*

Proof. First note that $\operatorname{Ext}^i_R(-, M)$ applied to $R/I^{n+2} \to R/I^{n+1} \to R/I^n \to \cdots$ gives the directed system $\operatorname{Ext}^i_R(R/I^n, M) \to \operatorname{Ext}^i_R(R/I^{n+1}, M) \to \operatorname{Ext}^i_R(R/I^{n+2}, M) \to \cdots$. In the $i = 0$ case, we have $\operatorname{Hom}_R(R/I^n, M) \cong (0 :_M I^n)$. So $\varinjlim \operatorname{Hom}_R(R/I^n, M) \cong \varinjlim (0 :_M I^n) \cong \cup_n (0 :_M I^n) = H^0_I(M)$. In general, let $E^{\cdot}$ be an injective resolution of M. Then, as $\varinjlim$ is exact, we have

$$\begin{aligned}
\varinjlim \operatorname{Ext}^i_R(R/I^n, M) &= \varinjlim H^i(\operatorname{Hom}_R(R/I^n, E^{\cdot})) \\
&\cong H^i(\varinjlim \operatorname{Hom}_R(R/I^n, E^{\cdot})) \\
&\cong H^i(H^0_I(E^{\cdot})) \\
&= H^i_I(M).
\end{aligned}$$

□

Definition. *Let $\underline{x} = x_1, ..., x_n \in R$. Define the Koszul co-complex on R with respect to $\underline{x}$ as follows:*

$$\begin{aligned}
n = 1: \quad K^{\cdot}(x_1; R) &:= 0 \to \overset{0}{R} \xrightarrow{x_1} \overset{1}{R} \to 0 \\
n > 1: \quad K^{\cdot}(\underrightarrow{x}; R) &:= K^{\cdot}(x_1, ..., x_{n-1}; R) \otimes K^{\cdot}(x_n; R) \\
&= \otimes_{i=1}^n K^{\cdot}(x_i; R)
\end{aligned}$$

which looks like

$$0 \to \overset{0}{R} \xrightarrow{1 \mapsto (x_1, ..., x_n)} \overset{1}{R} \to R^{\binom{n}{2}} \to \cdots \to R^n \xrightarrow{e_i \mapsto \pm x_i} \overset{n}{R} \to 0$$

This is essentially the same as $K_{\cdot}(\underline{x}; R)$, the Koszul complex, except it is written as a co-complex and the signs in the maps differ. If M is an $R-$module, define the Koszul co-complex on M with respect to $\underrightarrow{x}$ by $K^{\cdot}(\underline{x}; M) = K^{\cdot}(\underline{x}; R) \otimes_R M$. Then i^{th} **Koszul cohomology** on M with respect to $\underline{x}$ is $H^i(\underline{x}; M) = H^i(K^{\cdot}(\underline{x}; M))$.

Proposition 6.2. *Let $\underline{x} = x_1, ..., x_n \in R$, M an $R-$module. Then*

1. $H^0(\underline{x}; M) \cong (0 :_M (\underline{x}))$.
2. $H^n(\underline{x}; M) \cong M/(\underline{x})M$.
3. *If $x_1, ..., x_n$ is an $M-$regular sequence, then $H^i(\underline{x}; M) = 0$ for all $i < n$.*

Definition. *Let $M = \{M_\alpha\}, N = \{N_\alpha\}$ be directed systems of $R-$modules. Define a directed system $M \otimes_R N$ by $(M \otimes_R N)_\alpha = M_\alpha \otimes N_\alpha$ and $M_\alpha \otimes N_\alpha \xrightarrow{M^\alpha_\beta \otimes N^\alpha_\beta} M_\beta \otimes N_\beta$ for $\alpha \leq \beta$.*

Lemma 6.3. $\varinjlim (M_\alpha \otimes N_\alpha) \cong \varinjlim M_\alpha \otimes \varinjlim N_\alpha$.

Definition. *Let $\{C^{\cdot}_\alpha\}$ be a directed system of co-complexes of $R-$modules, that is,*

$$\begin{tikzcd}
\cdots \arrow[r] & C^n_\alpha \arrow[r] \arrow[d] & C^{n+1}_\alpha \arrow[r] \arrow[d] & C^{n+2}_\alpha \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & C^n_\beta \arrow[r] & C^{n+1}_\beta \arrow[r] & C^{n+2}_\beta \arrow[r] & \cdots
\end{tikzcd}$$

for $\alpha \leq \beta$. Then $\varinjlim C^{\cdot}_\alpha$ is a co-complex:

$$\cdots \to \varinjlim C^n_\alpha \to \varinjlim C^{n+1}_\alpha \to \varinjlim C^{n+2}_\alpha \to \cdots.$$

Definition. *Let $C^{\cdot}, D^{\cdot}$ be directed systems of co-complexes of $R-$modules. Define a directed system $C^{\cdot} \otimes_R D^i$ by*

$$\begin{tikzcd}
(C^{\cdot} \otimes_R D^{\cdot})^n_\alpha \arrow[r, equal] \arrow[d] & \sum_{i+j=n} C^i_\alpha \otimes D^j_\alpha \arrow[d] \\
(C \otimes_R D)^{n+1}_\beta \arrow[r, equal] & \sum_{i+j=n} C^i_\beta \otimes D^j_\beta
\end{tikzcd}$$

Fact. $\varinjlim (C^\cdot \otimes D^\cdot)_\alpha \cong (\varinjlim C^\cdot_\alpha) \otimes (\varinjlim D^\cdot_\alpha)$.

Recall for $x \in R$ that $\varinjlim (R \xrightarrow{x} R \xrightarrow{x} R \xrightarrow{x} \cdots) \cong R_x$. As a corollary to this, one can prove $\varinjlim (M \xrightarrow{x} M \xrightarrow{x} M \xrightarrow{x} \cdots) \cong M_x$.

Definition. *Let $\underline{x} = x_1, ..., x_n \in R$, M an $R-$module. Define a directed system $K^\cdot(\underline{x}^t; M)$ as follows:*

$$n = 1 \quad K^\cdot(x^t; M) := \quad \begin{tikzcd} 0 \arrow[r] & M \arrow[r, "x"] \arrow[d, "="] & M \arrow[r] \arrow[d, "x"] & 0 \\ 0 \arrow[r] & M \arrow[r, "x^2"] \arrow[d, "="] & M \arrow[r] \arrow[d, "x"] & 0 \\ 0 \arrow[r] & M \arrow[r, "x^3"] \arrow[d, "="] & M \arrow[r] \arrow[d, "="] & 0 \\ & \vdots & \vdots & \end{tikzcd}$$

$$n > 1 \quad K^\cdot(\underline{x}^t; M) := \quad K^\cdot(x_1^t, ..., x_{n-1}^t; M) \otimes K^\cdot(x_n^t; R)$$

Theorem 6.4. $\varinjlim K^\cdot(\underline{x}^t; M) \cong C^\cdot(\underline{x}; M)$*, the Čech Complex.*

Proof. We will prove by induction. Let $n = 1$. Clearly $\varinjlim (M \xrightarrow{=} M \xrightarrow{=} M \xrightarrow{=} \cdots) \cong M$. By the Corollary, $\varinjlim (M \xrightarrow{x} M \xrightarrow{x} M \xrightarrow{x} \cdots) \cong M_x$. One easily checks that the induced map on direct limits is $M \to M_x$ defined by $m \mapsto \frac{m}{1}$. So suppose $n > 1$. Then

$$\begin{aligned} \varinjlim K^\cdot(\underline{x}^t; M) &= \varinjlim (K^\cdot(x_1^t, ..., x_{n-1}^t; M) \otimes_R K^\cdot(x_n^t; R) \\ &= (\varinjlim K^\cdot(x_1^t, ..., x_{n-1}^t; M)) \otimes_R (\varinjlim K^\cdot(x_n^t; R)) \\ &= C^\cdot(x_1, ..., x_{n-1}; M) \otimes C^\cdot(x_n; R) \\ &= C^\cdot(\underline{x}; M). \end{aligned}$$

□

Theorem 6.5. *Let R be Noetherian, $I = (\underline{x})R$, M an $R-$module. Then $H_I^i(M) \cong \varinjlim H^i(\underline{x}^t; M)$.*

Proof. As $\varinjlim$ is exact,

$$\begin{aligned} H_I^i(M) &\cong H_{\underline{x}}^i(M) \\ &\cong H^i(C^\cdot(\underline{x}; M)) \\ &\cong H^i(\varinjlim K^\cdot(\underline{x}^t; M)) \\ &\cong \varinjlim H^i(K^\cdot(\underline{x}^t; M)) \\ &= \varinjlim H^i(\underline{x}^t; M). \end{aligned}$$

□

Corollary 6.6. *Let R be Noetherian, $I = (x_1, ..., x_n)R$, M an $R-$module. Then $H_I^n(M) \cong \varinjlim M/(x_1^t, ..., x_n^t)M$ where $M/(x_1^t, ..., x_n^t)M \xrightarrow{x_1 \cdots x_n} M/(x_1^{t+1}, ..., x_n^{t+1})M$.*

Remark. Let $\{I_n\}, \{J_n\}$ be two decreasing chains of ideals. We say the chains are **cofinal** if for all n there exists k such that $J_k \subseteq I_n$, and for all m there exists ℓ such that $I_\ell \subseteq J_m$.

If $\{I_n\}$ is a descending chain of ideals cofinal with $\{I^n\}$ then

$$H_I^0(M) = \cup_n (0 :_M I_n) = \varinjlim \operatorname{Hom}_R(R/I_n, M).$$

One can show that $H_I^i(M) = \varinjlim \operatorname{Ext}_R^i(R/I_n, M)$.

Theorem 6.7 (**Mayer-Vietoris sequence**). *Let R be a Noetherian ring, $I, J \subseteq R$, M an $R-$module. Then there exists a natural long exact sequence*

$$0 \to H^0_{I+J}(M) \to H^0_I(M) \oplus H^0_J(M) \to H^0_{I\cap J}(M) \to \cdots \to H^i_{I+J}(M) \to H^i_I(M) \oplus H^i_J(M) \to H^i_{I\cap J}(M) \to \cdots .$$

Proof. For all n there exists a short exact sequence

$$0 \to R/(I^n \cap J^n) \to R/I^n \oplus R/J^n \to R/(I^n + J^n) \to 0.$$

Apply $\mathrm{Hom}_R(-, M)$ to get a long exact sequence

$$\cdots \to \mathrm{Ext}^i_R(R/(I^n + J^n), M) \to \mathrm{Ext}^i_R(R/I^n \oplus R/J^n, M) \to \mathrm{Ext}^i_R(R/(I^n \cap J^n), M) \to \cdots .$$

This forms a directed system of long exact sequences. Take direct limits. It is enough to show $\{I^n + J^n\}$ is confinal with $\{(I + J)^n\}$ and $\{I^n \cap J^n\}$ is cofinal with $\{(I \cap J)^n\}$. We know $I^n + J^n \subseteq (I + J)^n$ and $(I + J)^{2n} \subseteq I^n + J^n$. Now $(I \cap J)^n \subseteq I^n \cap J^n$. By the Artin Rees Lemma, there exists $k = k(n)$ such that for all $m \geq k$

$$I^m \cap J^n = I^{m-k}(I^k \cap J^n) \subseteq I^{m-k} J^n.$$

Therefore, for $m \geq n + k$ we have

$$I^m \cap J^m \subseteq I^m \cap J^n \subseteq I^{m-k} J^n \subseteq I^n J^n \subseteq (I \cap J)^n.$$

□

Proposition 6.8 (Hartshorne). *Let (R, m) be a local ring such that* $\operatorname{depth} R \geq 2$. *Then* $U = \operatorname{Spec} R - \{m\}$ *is connected.*

Proof. Assume U is disconnected. Then there exist clopen sets $V(I) \cap U \neq \emptyset$ and $V(J) \cap U \neq \emptyset$ such that

$$(V(I) \cap U) \cup (V(J) \cap U) = U \text{ and } V(I) \cap V(J) \cap U = \emptyset.$$

Notice that the first is true if and only if $\sqrt{I \cap J} \subseteq \cup_{p \in \operatorname{Spec} R \setminus \{m\}} p = \sqrt{0}$ which is if and only if $I \cap J$ is nilpotent. The second equality is true if and only if $\sqrt{I + J} = m$ as I and J must be proper. Together with $V(I) \cap U \neq \emptyset$ and $V(J) \cap U \neq \emptyset$, we have neighther I nor J is $m-$primary or nilpotent.

By Mayer-Vietoris,

$$0 \to H^0_{I+J}(R) \to H^0_I(R) \oplus H^0_J(R) \to H^0_{I\cap J}(R) \to H^1_{I+J}(R).$$

Now $\sqrt{I + J} = m$ and $\operatorname{depth} R \geq 2$, so $H^0_{I+J}(R) = H^1_{I+J} = 0$. Also $H^0_{I\cap J}(R) = R$ as $I \cap J$ is nilpotent. Therefore $R \cong H^0_I(R) \oplus H^0_J(R)$. As R is local, R is indecomposable. Say $H^0_I(R) \cong R$, which implies $H^0_I(R)$ is generated by a nonzero-divisor. Thus I is nilpotent, a contradiction. □

7. Local Duality

Lemma 7.1 (**Flat Resolution Lemma**). *Let R be a ring, M, N $R-$modules and $F.$ a flat resolution of M, that is, each F_i is a flat $R-$module and $\cdots \to F_2 \to F_1 \to F_0 \to M \to 0$ is exact. Then $\mathrm{Tor}^R_i(M, N) \cong H_i(F. \otimes_R N)$ for all $i \geq 0$.*

Proof. Induct on i. For $i = 0$, as $- \otimes_R N$ is right exact we have $F_1 \otimes_R N \to F_0 \otimes_R N \to M \otimes_R N \to 0$ is exact. Thus $H_0(F. \otimes_R N) = M \otimes_R N = \mathrm{Tor}^R_0(M, N)$. Now suppose $i > 0$. Let $K_0 = \ker(F_0 \to M)$. Then $0 \to K_0 \to F_0 \to M \to 0$ is exact. As F_0 is flat, $\mathrm{Tor}^R_i(F_0, N) = 0$ for all $i \geq 1$. Therefore

$$0 \to \mathrm{Tor}^R_1(M, N) \to K_0 \otimes_R N \to F_0 \otimes_R N \to M \otimes_R N \to 0$$

is exact and $\mathrm{Tor}^R_i(M, N) \cong \mathrm{Tor}^R_{i-1}(K_0, N)$ for all $i \geq 2$.

For $i=1$ we have $\operatorname{Tor}_1^R(M,N)=\ker(K_0\otimes N\to F_0\otimes N)$ but from the diagram

$$\begin{tikzcd}
F_2\otimes N \arrow[r] & F_1\otimes N \arrow[rr] \arrow[dr] & & F_0\otimes N \\
& & K_0\otimes N \arrow[ur] \arrow[dr, "onto"] & \\
& & & 0
\end{tikzcd}$$

where the bottom sequence is exact we have

$$\ker(K_0\otimes N\to F_0\otimes N)\cong \ker(F_1\otimes N/\operatorname{im}(F_2\otimes N)\to F_0\otimes N)=H_1(F_1\otimes_R N).$$

For $i>1$ use the isomorphism $\operatorname{Tor}_i^R(M,N)\cong \operatorname{Tor}_{i-1}^R(K_0,N)$ for all $i\geq 2$ and the fact that $\cdots\to F_2\to F_1\to K_0\to 0$ is a flat resolution of K_0. $\square$

Theorem 7.2 (**Local Duality**). *Let (R,m) be a complete Cohen Macaulay local ring of dimension d. Then for all finitely generated $R-$modules M,*

$$\operatorname{Ext}_R^{d-i}(M,\omega_R)\cong H_m^i(M)^\vee \text{ and } \operatorname{Ext}_R^{d-i}(M,\omega_R)^\vee\cong H_m^i(M)$$

for all i where $(-)^\vee=\operatorname{Hom}_R(-,E_R(R/m))$.

Proof. We will prove the first isomorphism. The second isomorphism follows using Matlis Duality as $\operatorname{Ext}_R^{d-i}(M,\omega_R)$ is finitely generated and $H_m^i(M)$ is Artinian.

Let $x_1,...,x_d$ be a system of parameters for R. Then $C^\cdot(\underline{x};R)$ looks like $0\to R\to\oplus R_{x_i}\to\cdots\to R_{x_1\cdots x_d}\to 0$. The homology at the i^{th} place is $H^i_{(\underline{x})}(R)=H_m^i(R)$. As R is Cohen Macaulay, $H_m^i(R)=0$ for all $i<d$. Therefore

$$0\to R\to\oplus R_{x_i}\to\cdots\to R_{x_1\cdots x_d}\to H_m^d(R)\to 0$$

is exact. Hence $F.=C^\cdot(\underline{x};R)$ is a flat resolution of $H_m^d(R)$ (by letting $F_i=C^{d-i}$). Now

$$H_m^i(M)=H^i(C^\cdot(\underline{x};R)\otimes_R M)=H_{d-i}(F.\otimes_R M)\cong \operatorname{Tor}_{d-i}^R(H_m^d(R),M).$$

Computing this Tor using a free resolution $G.$ of M, we see $H_m^i(M)=H_{d-i}(G.\otimes_R H_m^d(R))$. Therefore, for all i, we have

$$\begin{aligned}
H_m^i(M)^\vee &= H_{d-i}(G.\otimes_R H_m^d(R))^\vee\\
&\cong H^{d-i}((G.\otimes_R H_m^d(R))^\vee) \text{ as } (-)^\vee \text{ is exact}\\
&\cong H^{d-i}(\operatorname{Hom}_R(G.\otimes H_m^d(R),E))\\
&\cong H^{d-i}(\operatorname{Hom}_R(G.,H_m^d(R)^\vee)) \text{ by Hom-}\otimes\text{ adjointness}\\
&\cong \operatorname{Ext}_R^{d-i}(M,H_m^d(R)^\vee)).
\end{aligned}$$

It is enough to show $\omega_R\cong H_m^d(R)^\vee$. Note $H_m^d(R)^\vee$ is finitely generated by Matlis Duality. Since our above isomorphism is true for i, we see $\operatorname{Ext}_R^i(M,H_m^d(R)^\vee)=0$ for $i>d$ and all finite $R-$modules M. Hence $\operatorname{Ext}_R^i(R/p,H_m^d(R)^\vee)=0$ for all $p\in\operatorname{Spec} R$ for $i>d$ which implies $\mu_i(p,H_m^d(R)^\vee)=0$ for all $p\in\operatorname{Spec}R$ for $i>d$. Thus $\operatorname{id}_R H_m^d(R)^\vee<\infty$. Also

$$\operatorname{Ext}_R^i(R/m,H_m^d(R)^\vee)=H_m^{d-i}(R/m)^\vee=\begin{cases}0 & \text{if } 0\leq i<d\\ R/m & \text{if } i=d\end{cases}.$$

Thus $\operatorname{depth} H_m^d(R)^\vee=d$ and $\mu_d(H_m^d(R)^\vee)=1$. Hence $\omega_R\cong H_m^d(R)^\vee$. $\square$

Remarks. Let (R,m) be a local ring and M an $R-$module. Let $\hat{R}$ denote the $m-$adic completion of R and $E=E_R(R/m)=E_{\hat{R}}(\hat{R}/\hat{m})$.

(1) $\operatorname{Hom}_{\hat{R}}(M\otimes_R\hat{R},E)\cong\operatorname{Hom}_R(M,E)$.

Proof. By Hom-$\otimes$ adjointness, $\operatorname{Hom}_{\hat{R}}(M \otimes_R \hat{R}, E) \cong \operatorname{Hom}_R(M, \operatorname{Hom}_{\hat{R}}(\hat{R}, E)) \cong \operatorname{Hom}_R(M, E)$. $\square$

(2) If M is Artinian then M is naturally an $\hat{R}-$module and $M \otimes_R \hat{R} \cong M$.
(3) If M is a finitely generated $R-$module $H^i_m(M) \cong H^i_{m\hat{R}}(\hat{M})$ for all i.

Proof. Note $H^i_{m\hat{R}}(\hat{M}) = H^i_m(M) \otimes_R \hat{R}$ and $H^i_m(M)$ is Artinian. $\square$

Theorem 7.3 (Version of Local Duality for Non-Complete Rings). *Let (R, m) be a $d-$dimensional Cohen Macaulay ring which is the homomorphic image of a Gorenstein ring. Let ω_R be the canonical module of R. Then for all finitely generated $R-$modules M and all i, $\operatorname{Ext}^{d-i}_R(M, \omega_R)^\vee \cong H^i_m(M)$.*

Proof.

$$\begin{aligned}
\operatorname{Ext}^{d-i}_R(M, \omega_R)^\vee) &= \operatorname{Hom}_R(\operatorname{Ext}^{d-i}_R(M, \omega_R), E) \\
&\cong \operatorname{Hom}_{\hat{R}}(\operatorname{Ext}^{d-i}_R(M, \omega_R) \otimes_R \hat{R}, E) \text{ by Remark 1} \\
&\cong \operatorname{Hom}_{\hat{R}}(\operatorname{Ext}^{d-i}_{\hat{R}}(\hat{M}, \omega_{\hat{R}}), E) \text{ as } \hat{\omega_R} = \omega_{\hat{R}} \\
&\cong H^i_{m\hat{R}}(\hat{M}) \text{ by the complete case of local duality} \\
&\cong H^i_m(M) \text{ by Remark 3.}
\end{aligned}$$

$\square$

Remark. Let (R, m) be a local Cohen Macaulay ring which has a canonical module. Let K be a finitely generated $R-$module. If $hatK \cong \hat{\omega_R}$ then $K \cong \omega_R$.

Proof. See Bruns and Herzog Proposition 3.3.14. $\square$

Proposition 7.4. *Let (R, m) be a Cohen Macaulay local ring which has a canonical module. Write $R \cong S/I$ where (S, n) is a Gorenstein local ring and $\operatorname{ht} I = g$. Then $\omega_R \cong \operatorname{Ext}^g_S(R, S)$.*

Proof. By the remark, it is enough to show $\operatorname{Ext}^g_S(R, S) \otimes_R \hat{R} \cong \omega_{\hat{R}} = H^d_{m\hat{R}}(\hat{R})^\vee$. Thus we may assume R and S are complete. Now

$$\begin{aligned}
\operatorname{Ext}^g_S(R, S)^\vee &= \operatorname{Hom}_R(\operatorname{Ext}^g_S(R, S), E_R(k)) \\
&= \operatorname{Hom}_R(\operatorname{Ext}^g_S(R, S), \operatorname{Hom}_S(R, E_S(k))) \\
&= \operatorname{Hom}_S(\operatorname{Ext}^g_S(R, S) \otimes_R R, E_S(k)) \\
&= \operatorname{Hom}_S(\operatorname{Ext}^g_S(R, S), E_S(k)) \\
&= H^{\dim S - g}_n(R) \text{ by local duality and as } \omega_S \cong S \\
&= H^{\dim R}_m(R) \text{ by the chance of rings principal}
\end{aligned}$$

By Matlis Duality, $\operatorname{Ext}^g_S(R, S) \cong H^{\dim R}_m(R)^\vee \cong \omega_R$. $\square$

Theorem 7.5 (Chevelley's Theorem). *Let (R, m) be a complete local ring. If I_n for $n = 1, 2, ..$ are ideals of R such that $I_n \supseteq I_{n+1}$ for all n and $\cap_n I_n = 0$ then for any $n \in \mathbb{N}$ there exists $s = s(n) \in \mathbb{N}$ such that $I_s \subseteq m^n$.*

Proof. We will prove by contradiction. Assume there exists $r \in \mathbb{N}$ such that $I_s \nsubseteq m^r$ for any $s \in \mathbb{N}$. Then for any $n \geq r$, $I_s \nsubseteq m^n$ for all s. Now $\dim R/m^n = 0$ and so R/m^n is Artinian. Thus there exists $t(n) \in \mathbb{N}$ such that $I_{t(n)} + m^n = I_s + m^n$ for all $s > t(n)$. Now we may assume $t(n) < t(n+1)$ for any $n > r$. Then $I_{t(n)} \subseteq I_{t(n)} + m^n = I_{t(n+1)} + m^n$. Therefore for any $x_n \in I_{t(n)}$ there exists $x_{n+1} \in I_{t(n+1)}$ such that $x_n - x_{n+1} \in m^n$. Start with $x_r \in I_{t(r)} \setminus m^r$. Then we have a sequence $(x_n)_{n \geq r}$ such that $x_n - x_{n+1} \in m^n$. Clearly, (x_n) is a Cauchy sequence. As R is complete, let $x^* = \lim_{n \to \infty} x_n$. Now $x_n, x_{n+1}, ... \in I_{t(n)}$. As ideals are closed in the $m-$adic topology $x^* \in I_{t(n)}$ and so $x^* \in \cap_{n \geq r} I_{t(n)} = 0$.

On the other hand, $x_n - x_r \in m^r$ for all $n \geq r$. So $x^* - x_r \in m^r$ (as there exists $n \geq r$ such that $x^* - x_n \in m^r$ and so $(x^* - x_n) + (x_n - x_r) \in m^r$). Thus $x_r \in m^r$, a contradiction. $\square$

Theorem 7.6. *Let (R, m) be a local ring and M a finite $R-$module of dimension s. Then $H^s_m(M) \neq 0$. Hence $\dim M = \sup\{i | H^i_m(M) \neq 0\}$.*

Proof. Since $\dim \hat{M} = \dim M$ and $H^i_{\hat{m}}(\hat{M}) \cong H^i_m(M)$, we may assume R is complete. Let $R = S/I$ where (S, n) is a complete regular local ring. By the change of rings principle, it is enough to show $H^s_n(M) \neq 0$ where M is considered as an $S-$module. Let $g = \operatorname{ht} \operatorname{Ann}_S M$. As S is Cohen Macaulay, there exists $x_1, .., x_g \in \operatorname{Ann}_S M$ which form an $S-$sequence. Let $T = S/(x_1, ..., x_g)$. Then (T, n_1) is a complete Gorenstein local ring, M is a finite $T-$module, and $\dim M = \dim T = S$. By the change of rings principle, it is enough to show $H^s_{n_1}(M) \neq 0$ where M is considered as a $T-$module. $\square$

Definition. *Let (R, m) be a local ring and M a finitely generated $R-$module. M is said to be a **Buchsbaum module** if and only if for all system of parameters$\underline{x} = x_1, ..., x_r \in R$ for M (that is, $r = \dim M$ and $\lambda(M/(\underline{x})M) < \infty$),*

$$\lambda(M/(\underline{x})M) - e_{(\underline{x})}(M) = C, \text{ a constant}$$

Recall $e_{(\underline{x})}(M) = \lim_{n\to\infty} \frac{\lambda(M/(x)^n M)}{n^r} \cdot r!$, the multiplicity of M with respect to $(\underline{x})$.

Note. Since $e_{(\underline{x})}(M) = \lambda(M/(\underline{x})M)$ if $\underline{x}$ is an $M-$sequence, Cohen Macaulay modules are Buchsbaum.

Theorem 7.7 (Stückrad-Vogel)**.** *If M is a Buchsbaum module of dimension d, then $m \cdot H^i_m(M) = 0$ for all $i < d$. The converse, however, does not hold. (There is no known cohomological characterization of Buchsbaum modules).*

Note that as $H^i_m(M)$ are Artinian $R/m-$modules, this means $\dim_{R/m} H^i_m(M) < \infty$ for all $i < d$. This lead to the following.

Definition. *Let (R, m) be a local ring and M a finitely generated $R-$module. M is said to be a **generalized Cohen Macaulay module** if $\lambda(H^i_m(M)) < \infty$ for all $i < \dim M$.*

Remark. Buchsbaum modules are generalized Cohen-Macaulay modules. Let

$$I(M) := \sup_{\underline{x} \in R, s.o.p\ for\ M} \{\lambda(M/(\underline{x})M) - e_{(\underline{x})}(M)\}.$$

Theorem 7.8 (Cuong-Schezel-Trun, 1978)**.** *Let (R, m) be a local ring and M a finite $R-$module. TFAE*

(1) *M is generalized Cohen Macaulay.*
(2) *$I(M) < \infty$.*

Moreover, if either holds then $I(M) = \sum_{i=0}^{d-1} \binom{d-1}{i} \lambda(H^i_m(M))$ for $d = \dim M$.

Definition. *A finite $R-$module M is **equidimensional** if $\dim R/p = \dim M$ for all $p \in \operatorname{Min}_R M = \operatorname{Min}_R(R/\operatorname{Ann}_R M)$, that is, $R/\operatorname{Ann}_R M$ is equidimensional.*

Remark. We always have $\dim R/p + \dim M_p \leq \dim M$ for all $p \supseteq \operatorname{Ann}_R M$. If R is local and catenary, then M is equidimensional if and only if $\dim R/p + \dim M_p = \dim M$ for all $p \supseteq \operatorname{Ann}_R M$.

Lemma 7.9. *Let (R, m) be a local ring and N an $R-$module. Then $\operatorname{Ann}_R N = \operatorname{Ann}_R N^\vee$.*

Proof. Certainly $\operatorname{Ann}_R N \subseteq \operatorname{Ann}_R \operatorname{Hom}_R(N, E) = \operatorname{Ann}_R N^\vee$. Thus $\operatorname{Ann}_R N^\vee \subseteq \operatorname{Ann}_R N^{\vee\vee}$. But the natural map $N \to N^{\vee\vee}$ is always injective, so $\operatorname{Ann}_R N^{\vee\vee} \subseteq \operatorname{Ann}_R N$ implies $\operatorname{Ann}_R N^\vee \subseteq \operatorname{Ann}_R N$. $\square$

Theorem 7.10. *Let (R, m) be a local ring which is the homomorphic image of a Gorenstein ring. Let M be a finite $R-$module. TFAE*

(1) *M is generalized Cohen Macaulay.*
(2) *M is equidimensional and M_p is Cohen Macaulay for all $p \in \operatorname{Spec} R \setminus \{m\}$.*

Proof. Let $R = S/I$ where (S, n) is a local Gorenstein ring. Then M is an $S-$module in the natural way. By the change of rings principle, $H^i_n(M) \cong H^i_m(M)$ for all i (where M is considered as an $S-$module on the left hand side and as an $R-$module on the right hand side). Therefore, M is generalized Cohen Macaulay as an $R-$module if and only if it is as an $S-$module. Likewise, M is equidimensional as an $R-$module if and only if it is as an $S-$module (since $S/\operatorname{Ann}_S M = R/\operatorname{Ann}_R M$) and M_q is Cohen Macaulay for all $q \in \operatorname{Spec} S \setminus \{m\}$ if and only if M_p is Cohen Macaulay for all $p \in \operatorname{Spec} R \setminus \{m\}$. Thus we may assume (R, m) is Gorenstein.

Note that as $H^i_m(M)$ is Artinian, $\lambda(H^i_m(M)) < \infty$ if and only if $m^n H^i_m(M) = 0$ for some n if and only if $m^n \subseteq \operatorname{Ann}_R H^i_m(M)$ for some n. By local duality, $H^i_m(M) = \operatorname{Ext}^{d-i}_R(M, R)^\vee$. By the Lemma, $\operatorname{Ann}_R H^i_m M) = \operatorname{Ann}_R \operatorname{Ext}^{d-i}_R(M, R)$. Thus

$$\begin{aligned} \lambda(H^i_m(M)) < \infty \quad &\Leftrightarrow \quad m^n \subseteq \operatorname{Ann}_R \operatorname{Ext}^{d-i}_R(M, R) \\ &\Leftrightarrow \quad \operatorname{Ext}^{d-i}_R(M, R)_p = 0 \text{ for all } p \neq m \text{ as } \operatorname{Ext}^{d-i}_R(M, R) \text{ is finitely generated} \\ &\Leftrightarrow \quad \operatorname{Ext}^{d-i}_{R_p}(M_p, R_p) = 0 \text{ for all } p \neq m, p \supseteq \operatorname{Ann}_R M. \end{aligned}$$

As R_p is Gorenstein, we can use local duality again to say $\operatorname{Ext}^{d-i}_{R_p}(M_p, R_p)^\vee \cong H^{\operatorname{ht}(p)-(d-i)}_{pR_p}(M_p)$. Thus (as $N = 0$ if and only if $N^\vee = 0$), we see $\operatorname{Ext}^{d-i}_{R_p}(M_p, R_p) = 0$ if and only if $H^{\operatorname{ht}(p)-d+i}_{pR_p}(M_p) = 0$. Thus we arrive at the following

$$(*)\ \lambda(H^i_m(M)) < \infty \Leftrightarrow H^{i-\dim R/p}_{pR_p}(M_p) = 0 \text{ for all } p \neq m, p \supseteq \operatorname{Ann}_R M.$$

For (2) $\Rightarrow$ (1), as M_p is Cohen Macaulay for all $p \neq m$, $H^{i-\dim R/p}_{pR_p}(M_p) = 0$ for all $i - \dim R/p < \dim M_p$, which implies $H^{i-\dim R/p}_{pR_p}(M_p) = 0$ for all $i < \dim M$ by the Remark. Therefore $\lambda(H^i_m(M)) < \infty$ for all $i < \dim M$.

For (1) $\Rightarrow$ (2) $H^{i-\dim R/p}_{pR_p}(M_p) = 0$ for all $i < \dim M$ and for all $p \neq m$ with $p \supseteq \operatorname{Ann}_R M$, or, $H^j_{pR_p}(M_p) = 0$ for all $j < \dim M - \dim R/p$ and for all $p \neq m$ with $p \supseteq \operatorname{Ann}_R M$. Since $H^{\dim M_p}_{pR_p}(M_p) \neq 0$, this says that $\dim M_p \geq \dim M - \dim R/p$ for all $p \neq m, p \supseteq \operatorname{Ann}_R M$. Since we always have $\dim M_p \leq \dim M - \dim R/p$ for all $p \supseteq \operatorname{Ann}_R M$, we have $\dim M_p = \dim M - \dim R/p$ for all $p \neq m, p \supseteq \operatorname{Ann}_R M$. Thus M is equidimensional and $H^j_{pR_p}(M_p) = 0$ for all $j < \dim M_p$. So M_p is Cohen Macaulay for all $p \neq m$. □

Recall that $\operatorname{soc}(M) := (0 :_M m) = \{x \in M | mx = 0\}$.

Lemma 7.11. *Let (R, m) be a local ring and M a finitely generated $R-$module. Then $\mu(M) = \dim_{R/m} \operatorname{soc}(M^\vee)$.*

Proof. Since $\mu(M) = \mu(M^\vee)$ and $M^\vee \cong \hat{M}^\vee$, we may assume R is complete. Consider $0 \to mM \to M \to L \to 0$ where $\mu(M) = \dim_k L$ for $k = R/m$. Since $0 \to L^\vee \to M^\vee$ is exact and $m \cdot L^\vee = 0$, $\dim \operatorname{soc}(M^\vee) \geq \dim L^\vee = \mu(M)$. On the other hand, let $V = \operatorname{soc}(M^\vee)$. From $0 \to V \to M^\vee \to B \to 0$ we get $M^{\vee\vee} \to V^\vee \to 0$ is exact. As R is complete, $\mu(M) = \mu(M^{\vee\vee}) \geq \mu(V^\vee) = \dim V^\vee = \dim V$. □

<u>Question:</u> Let (R, m) be a local ring of dimension d and I an ideal of R. When is $H^d_I(R) = 0$?

Certainly we need $\sqrt{I} \neq m$. Is that enough? The Hartshorne-Lichtenbaum Vanishing Theorem (HLVT) answers this. A special case of HLVT is the following:

- Let (R, m) be a complete domain of dimension d. Then $H^d_I(R) = 0$ if and only if $\dim R/I > 0$ (that is, $\sqrt{I} \neq m$).

We will actually prove a more general version for arbitrary local rings. But first we begin with a very special case.

Proposition 7.12. *Let (R, m) be a complete local Gorenstein domain of dimension d. Let $p \in \operatorname{Spec} R$ with $\dim R/p = 1$. Then $H^d_p(R) = 0$.*

Proof. We first need to show the following claim.

Claim. $\{P^n\}_{n\geq 1}$ and $\{P^{(n)}\}_{n\geq 1}$ are cofinal.

Proof. As R is a domain $\cap_{n\geq 1} P^{(n)} = 0$ (Check). By Chevalley's Theorem for all k there exists n such that $P^{(n)} \subseteq m^k$. By primary decomposition $P^n = P^{(n)} \cap J_n$ where J_n is primary to m. Therefore $m^k \subseteq J_n$ for some k and so there exists $t >> 0$ such that $p^{(t)} \subseteq m^k \subseteq J_n$. We may as well assume $t \geq n$. Then $P^n = P^{(n)} \cap J_n \supseteq P^{(n)} \cap P^{(t)} = P^{(t)}$. Thus they are cofinal.

Note that $\operatorname{depth} R/P^{(n)} > 0$ for all n as $\operatorname{Ass}_R R/P^{(n)} = \{P\}$. Now $H^d_P(R) = \varinjlim \operatorname{Ext}^d_R(R/P^{(n)}, R)$. But by local duality $\operatorname{Ext}^d_R(R/P^{(n)}, R) = H^0_m(R/P^{(n)})^\vee = 0$. Thus $H^d_p(R) = 0$. □

Lemma 7.13. *Let R be a Noetherian ring, I an ideal, $x \in R$, and M an $R-$module. Then there exists a long exact sequence*

$$\cdots \to H^i_{(I,x)}(M) \to H^i_I(M) \to H^i_{I_x}(M_x) \to H^{i+1}_{(I,x)}(M) \to \cdots.$$

Proof. We proved this for Čech Cohomology earlier. $\square$

Proposition 7.14. *Let (R,m) be a local ring of dimension d. TFAE*

(1) $H^d_I(R) = 0$ *for all ideals I such that* $\dim R/I > 0$
(2) $H^d_p(R) = 0$ *for all $p \in \operatorname{Spec} R$ such that* $\dim R/p = 1$.

Proof. Clearly (1) implies (2). So suppose there exists an ideal I such that $\dim R/I > 0$ and $H^d_I(R) = 0$. Let I be maximal with respect to this property. By hypothesis, I is not prime of dimension 1. Thus there exists $x \in R \setminus I$ such that $\dim R/(I,x) > 0$. By the long exact sequence since $H^d_I(R) \neq 0$ and $H^d_{I_x}(R_x) = 0$ (as $\dim R_x < d$), we have $H^d_{(I,x)}(R) \neq 0$, a contradiction. $\square$

Proposition 7.15. *Let (R,m) be a local ring of dimension d, $I \subseteq R$ and M an $R-$module. Then $H^d_I(M) \cong H^d_I(R) \otimes_R M$. Hence if $H^d_I(R) = 0$ then $H^d_I(M) = 0$ for all $R-$modules M.*

Proof. As $\operatorname{ara}(I) \leq d$, let $I = \sqrt{(x_1, ..., x_d)}$ for some $x_1, ..., x_d \in R$. Then $\oplus_i R_{x_1 \cdots \hat{x_i} \cdots x_d} \to R_{x_1 \cdots x_d} \to H^d_I(R) \to 0$ is exact. Tensoring with M gives us $\oplus_i M_{x_1 \cdots \hat{x_i} \cdots x_d} \to M_{x_1 \cdots x_d} \to H^d_I(R) \otimes_R M \to 0$ is exact. But this implies $H^d_I(M) \cong H^d_I(R) \otimes_R M$. $\square$

Corollary 7.16. *Let (R,m) be a local ring of dimension d. TFAE*

(1) $H^d_I(R) = 0$
(2) $H^d_I(M) = 0$ *for all $R-$modules M.*

Let (R,m) be a local ring. Then one of the following holds:

(1) $\operatorname{char} R = 0$ and $\operatorname{char} R/m = 0$
(2) $\operatorname{char} R = p$ and $\operatorname{char} R/m = p$
(3) $\operatorname{char} R = 0$ and $\operatorname{char} R/m = p$
(4) $\operatorname{char} R = p^n, n > 1$ and $\operatorname{char} R/m = p$.

If (1) or (2) hold, R is said to have **equal characteristic**; otherwise, R has unequal characteristic. Note also that (1) holds if and only if $\mathbb{Q} \subseteq R$ and (2) holds if and only if $\mathbb{Z}_p \subseteq R$. Thus R has equal characteristic if and only if R contains a field.

Definition. *Let (R,m) be a complete local ring. A subring $K \subseteq R$ is called a* ***coefficient ring*** *for R if*

(1) $R = K + m$
(2) *If R has equal characteristic, then K is a field. Otherwise (K,n) is a complete local ring such that $n = pK$ where $p = \operatorname{char} R/m$.*

Note here that $R/m \cong K/n$. Also if R is a domain then K is a domain. Hence K is a field or a complete DVR. In any case, K is a quotient of a complete DVR.

Theorem 7.17 (Cohen)**.** *Every complete local ring has a coefficient ring.*

Proof. See Matsamura $\square$

Lemma 7.18. *Let (R,m) be a complete local ring, K a coefficient ring for R and $y_1, ..., y_d$ a system of parameters for R. Let $A = K[\![y_1, ..., y_d]\!]$. Then R is a finite $A-$module.*

Proof. First note that A is the image of the ring map $\phi : K[\![T_1, ..., T_d]\!] \to R$ defined by $T_i \mapsto y_i$. Therefore as $K[\![T_1, ..., T_d]\!]$ is complete and local, so is A. Let n be the maximal ideals of A. Then $n = (p, y_1, ..., y_d)A$ where $p = \operatorname{char} R/m$ (here p may be prime or 0). Clearly $n \subseteq m$. By definition of coefficient ring, $A/n \cong R/m$. Therefore every $R-$module of finite length has finite length as an $A-$module. In particular, $\lambda_A(R/nR) < \infty$ (as n contains a system of parameters for R). Choose $x_1, ..., x_r \in R$ such that $R/nR = A\overline{x_1} + ... + A\overline{x}_r$.

Claim. $R = Ax_1 + ... + Ax_r$.

Proof. We have $R = \sum Ax_i + nR$. Let $u \in R$. Write $u = \sum a_{i,0}x_i + u_1$ for $a_{i,0} \in A, u_1 \in n^R$ and iteratively $u_k = \sum a_{i,k}x_i + u_{k+1}$ for $a_{i,k} \in n^k, u_{k+1} \in n^{k+1}R$. Now for each i we have $a_i = a_{i,0} + a_{i,1} + ...$ converges in A. Then $u - \sum_{i=1}^r a_i x_i \in \cap n^k R \subseteq \cap m^k = 0$, a contradiction.

□

Proposition 7.19. *Let (R, m) be a complete local domain of dimension d and I an ideal of R. TFAE*

(1) $H_I^d(R) \neq 0$
(2) $\dim R/I = 0$.

Proof (due to Huneke and Brodmann, independently in 1994). The content of the proof is that (2) implies (1). By Proposition 7.14, it is enough to show $H_p^d(R) = 0$ for any $p \in \operatorname{Spec} R$ such that $\dim R/p = 1$. Let K be a coefficient ring for R. As R is a domain, K is a field or a complete DVR with uniformizing parameter q where $q = \operatorname{char} R/m$.

Let $p \in \operatorname{Spec} R$ with $\dim R/p = 1$. As $\operatorname{ara}(I) \leq d$, we know there exists $x_1, ..., x_d \in R$ such that $p = \sqrt{(x_1, ..., x_d)}$. Furthermore, we may choose $x_1, .., x_d$ with the following properties.

(1) $x_1, ..., x_{d-1}$ form part of a system of parameters for R as $\operatorname{ht} p = d - 1$.
(2) If K is not a field and $q \in p$, then $x_1 = q$ as R is a domain.
(3) If K is not a field and $q \notin p$, then $x_1, ..., x_{d-1}, q$ is a system of parameters for R (as $\sqrt{(p, q)} = m$, we may choose $\overline{x_1}, ..., \overline{x_{d-1}} \in \overline{p} = (p + q)/q$ to form a system of parameters for R/q).

If K is either a field or $q \in p$, choose $y \in R$ such that $x_1, ..., x_{d-1}, y$ is a system of parameters for R. If $q \notin p$, let $y = q$. By (3) $x_1, ..., x_{d-1}, y$ is a system of parameters for R.

Let $A = K[\![x_1, ..., x_{d-1}, y]\!]$. Then (as remarked in the previous lemma) A is a complete local domain as R is a domain and R is a finite $A-$module. Thus $\dim A = \dim R = d$.

Claim. A is a complete regular local ring.

Proof. First suppose K is a field. Then $A \cong K[\![T_1, ..., T_d]\!]/I$ where $T_1, ..., T_d$ are indeterminates. As $K[\![T_1, ..., T_d]\!]$ is a $d-$dimensional complete regular local ring and $\dim A = d$, $I = 0$.

Now suppose K is not a field. Then $q \in A$. Hence $A = K[\![x_2, ..., x_{d-1}, y]\!]$ if $x_1 = q$ or $A = [\![x_1, ..., x_{d-1}]\!]$ if $y = q$. In either case, $A \cong K[\![T_1, ..., t_d]\!]/I$. Again $K[\![T_1, ..., T_d]\!]$ is a complete regular local ring of dimension d and so $I = 0$.

Now let $B = A[x_d]$. Then $A \subseteq B \subseteq R$.

Claim. B is a complete local Gorenstein domain and R is a finite $B-$module.

Proof. As R is a finite $A-$module, R is certainly a finite $B-$module. Clearly B is Noetherian (as A is). Since R is a domain, so is B. As R is integral over B, any maximal ideal of B is contracted from R. As R is local, B must be also.

To see B is complete, first note that as B is a finite $A-$module and A is complete, B is complete as an $A-$module. Let m_A, m_B represent the maximal ideals of A and B respectively. As B/A is integral, $\sqrt{m_A B} = m_B$. Therefore $m_B^n \subseteq m_A B$ for some b. Hence, the m_A and m_B-adic topologies on B are equivalent and so B is complete.

Finally, $B = A[x_d] \cong A[T]/I$ where T is an indeterminate and I is a prime ideal. Since we know B is local, $B \cong A[t]_M/I_M$ where $M = (m_A, T)A[T]$. Now A is a regular local ring of dimension d

and so $A[T]_M$ is a regular local ring of dimension $d+1$. Since B is a domain of dimension d, I_M is a height 1 prime of $A[T]_M$ and hence principal (since a RLR is a UFD).

Now let $Q = P \cap B$. Since R/p is integral over B/Q, $\dim B/Q = 1$. By Proposition 7.12, $H^d_Q(B) = 0$. Since $P = \sqrt{(x_1, ..., x_d)}$ and $x_1, ..., x_d \in B$, $Q = \sqrt{(x_1, ..., x_d)B}$ (by the lying over theorem). Thus by change of rings and Proposition 7.15, we have

$$H^d_p(R) = H^d_{(x_1,...,x_d)R}(R) = H^d_{(x_1,...,x_d)B}(R) = H^d_{(x_1,...,x_d)B}(B) \otimes_B R = H^d_Q(B) \otimes_B R = 0.$$

□

Remarks. The proof given also shows that if (R, m) is a complete local domain of dimension d then there exists a complete regular local ring A of dimension d such that R is a finite $A-$module.

8. Hartshorne-Lictenbaum Vanishing Theorem

Theorem 8.1 (Hartshorne-Lichtenbaum Vanishing Theorem, 1968)**.** *Let (R, m) be a local ring of dimension d and I an ideal of R. TFAE*

(1) $H^d_I(R) = 0$
(2) $\dim \hat{R}/(I\hat{R} + p) > 0$ *for all* $p \in \operatorname{Spec} \hat{R}$ *such that* $\dim \hat{R}/p = d$.
(3) $H^d_I(M) = 0$ *for all $R-$modules M.*

Proof. We have already shown the equivalence of 1 and 3 (as a corollary to Proposition 7.15). We will show the equivalence of 1 and 2. Suppose $H^d_I(R) = 0$. Let $p \in \operatorname{Spec} \hat{R}$ such that $\dim \hat{R}/p = d$. Then $H^d_{(I\hat{R}+p)/p}(\hat{R}/p) \cong H^d_I(R) \otimes_R \hat{R}/p = 0$. By Proposition 7.19, we see $\dim \hat{R}/(I\hat{R} + p) > 0$.

For the other direction suppose $H^d_I(R) \neq 0$. Then $H^d_{I\hat{R}}(\hat{R}) \neq 0$ as $\hat{R}$ is a faithfully flat $R-$module. Let J be an ideal of $\hat{R}$ maximal with respect to the property that $H^d_{I\hat{R}}(\hat{R}/J) \neq 0$. Then $\dim \hat{R}/J = d$. Let $p \in \operatorname{Ass}_{\hat{R}}(\hat{R}/J)$ such that $\dim \hat{R}/p = d$. Then we have an exact sequence

$$0 \to \hat{R}/p \xrightarrow{\phi} \hat{R}/J \to \hat{R}/(J, x) \to 0$$

where $\phi(\overline{1}) = \overline{x} \neq 0$. Then

$$H^d_{I\hat{R}}(\hat{R}/p) \to \underbrace{H^d_{I\hat{R}}(\hat{R}/J)}_{\neq 0} \to \underbrace{H^d_{I\hat{R}}(\hat{R}/(J, x))}_{=0}$$

and so $H^d_{I\hat{R}}(\hat{R}/p) \neq 0$ by exactness, a contradiction. □

History. Originally, Lichtenbaum conjectured a geometric analogue of this vanishing theorem for sheaf cohomology. Grothendieck proved this conjecture in 1961 (nevertheless, it became known as "Lichtenbaum's Theorem"). Hartshorne proved this local vanishing theorem in 1968. Lichtenbaum's Theorem follows readily from Hartshorne's.

Theorem 8.2 (Faltings, 1979)**.** *Let (R, m) be a complete local domain of dimension d and I an ideal such that* $\operatorname{ara}(I) \leq d - 2$. *Then* $\operatorname{Spec}(R/I) - \{m/I\}$ *is connected.*

Proof. (due to J. Rung) Let $U = \operatorname{Spec}(R/I) \setminus \{m/I\} \cong V(I) \setminus \{m\}$. Suppose U is disconnected. This means there exist ideals $J, K \supseteq I$ in R such that

(1) $J \cap K \subseteq \sqrt{I}$ (and so $\sqrt{J \cap K} = \sqrt{I}$)
(2) $\sqrt{J + K} = m$
(3) $\sqrt{J} \neq m$ and $\sqrt{K} \neq m$ (that is, $\dim R/J, \dim R/K > 0$)

By the Mayer-Vietoris sequence, we have

$$H^{d-1}_{J+K}(R) \to H^{d-1}_J(R) \oplus H^{d-1}_K(R) \to H^{d-1}_{J\cap K}(R) \to H^d_{J+K}(R) \to H^d_J(R) \oplus H^d_K(R).$$

Now $H^{d-1}_{J\cap K}(R) = 0$ as $\sqrt{J\cap K} = \sqrt{I}$ and $\operatorname{ara}(I) \le d-2$. Thus $0 \to H^d_m(R) \to H^d_J(R) \oplus H^d_K(R)$ is exact. Since $H^d_m(R) \ne 0$ we have either $H^d_J(R) \ne 0$ or $H^d_K(R) \ne 0$. But $\dim R/J > 0$ and $\dim R/K > 0$, a contradiction to the HLVT. $\square$

This theorem has the following geometric consequence.

Theorem 8.3 (Fulton-Hansen, 1979)**.** *Let K be an algebraically closed field and X, Y irreductible projective varieties in $\mathbb{P}^n_k$. Suppose $\dim X + \dim Y > n$. Then $X \cap Y$ is connected.*

Idea of Proof. Use reduction to the diagonal: $K(X\times Y) = K(X)\otimes_K K(Y) \cong K[X_0, ..., X_n, Y_0, ..., Y_n]/I(X) + I(Y)$ has dimension $> n+2$. Now mod out by $\{X_i - Y_i\}_{i=0}^n$ and use Falting's result. $\square$

Question. Let (R, m) be a complete local domain, $I \subseteq R$. When is $H^{d-1}_I(R) = 0$ and $H^d_I(R) = 0$ for $d = \dim R$? One might guess it is if and only if $\dim R/I > 1$. But this is false, as shown by the following example of Hartshorne.

Example. Let $R = k[\![x, y, u, v]\!]/(xu - yv)$, where k is a field. Then R is a three-dimensional complete Gorenstein domain (in fact, it is a hypersurface). Let $I = (x, y)R$. Then $R/I \cong k[\![u, v]\!]$ and so I is a prime of dimension 2. If the conjecture were true, then $H^2_I(R) = 0$. We know $H^3_I(R) = 0$ as $\mu(I) = 2$. Let $J = (u, v)R$. Consider the short exact sequence $0 \to J \to R \to R/J \to 0$. Then $\cdots \to H^2_I(R) \to H^2_I(R/J) \to H^3_I(J) = 0$ is exact ($H^3_I(J) = 0$ as $\mu(I) = 2$). But $H^2_I(R/J) = H^2_{(I+J)/J}(R/J) = H^2_{m/J}(R/J) \ne 0$ as $\dim R/J = 2$. So $H^2_I(R) \ne 0$.

Note that in this example $\operatorname{ht} I = \operatorname{ht} J = 1$ but $\operatorname{ht}(I+J) = \operatorname{ht}(m) = 3$. If R is a regular local ring, we always have $\operatorname{ht}(p+q) \le \operatorname{ht} p + \operatorname{ht} q$ for all $p, q \in \operatorname{Spec} R$. Thus there is reason to believe the conjecture may hold for regular local rings.

Theorem 8.4 (Peskine-Szpiro in $\operatorname{char} p > 0$ (1973) and Ogus in $\operatorname{char} 0$ (1973))**.** *Let (R, m) be a complete regular local ring containing a field. Suppose R/m is algebraically closed. Let I be an ideal of R. TFAE*

(1) $H^{d-1}_I(R) = H^d_I(R) = 0$
(2) $\dim R/p > 1$ *for all* $p \in \operatorname{Min} R/I$ *and* $\operatorname{Spec}(R/I)\setminus\{m/I\}$ *is connected.*

Further improvements of the theorem have been given by Huneke and Lyubeznik.

Theorem 8.5 (Sharp, 1981)**.** *Let (R, m) be a local ring, I an ideal of R and M a finite $R-$module of dimension n. Then $H^n_I(M)$ is Artinian.*

Proof. As $R \to \hat{R}$is faithfully flat, if $H^n_{I\hat{R}}(\hat{M}) = H^n_I(M)\otimes_R \hat{R}$ has DCC, then $H^n_I(M)$ has DCC. Thus we may assume R is complete. By the change of rings principle, we may pass to the ring $R/\operatorname{Ann}_R M$ and so assume $\operatorname{Ann}_R M = 0$ and $\dim R = \dim M = n$.

Let $R = S/L$ where S is a complete regular local ring. Let $g = \operatorname{ht} L$ and $x_1, ..., x_g \in L$ an $S-$sequence. Let $B = S/(\underline{x})$ and $J = L/(\underline{x})$. Then $R = B/J$ where $\dim R = \dim B = n$ and B is a complete Gorenstein ring. Now M can be considered as a $B-$module. Thus it is enough to show $H^n_{IB}(M)$ is Artinian.

Claim. $H^n_J(B)$ is Artinian for any ideal J.
Proof. An injective resolution for B looks like

$$0 \to B \to \overset{0}{\underset{\operatorname{ht} p=0}{\oplus} E_B(B/p)} \to \cdots \to \overset{n}{E_B(B/m)} \to 0.$$

We know $E_B(B/m)$ is Artinian. Thus $\operatorname{Hom}_B(B/J, E)$ is Artinian. Now $H^n_J(B)$ is a quotient of this module and is hence Artinian.

Now we have seen $H^n_J(M) \cong H^n_J(B)\otimes_B M$ as $n = \dim B$. As $H^n_J(B)$ is Artinian, it is enough to show $N\otimes_B M$ is Artinian if N is Artinian and M is finitely generated. By Matlis Duality, it is enough to show $(N\otimes_B M)^\vee$ is finitely generated. But $(N\otimes_B M)^\vee = \operatorname{Hom}_B(N\otimes_B M, E) = \operatorname{Hom}_B(M, N^\vee)$ is finitely generated as $N^\vee$ is. $\square$

8.1. **An application of HLVT.**

Definition. *Let (R,m) be a local ring, M an $R-$module and $E = E_R(R/m)$. A* ***coassociated prime*** *of M is an associated prime of $M^\vee = \operatorname{Hom}_R(M,E)$. That is, $\operatorname{Coass}(M) = \operatorname{Ass}(M^\vee)$.*

Remarks.

(1) Let (R,m) be a local ring, M a finitely generated $R-$module, N any $R-$module. Then we have that $\operatorname{Ass}\operatorname{Hom}_R(M,N) = \operatorname{Supp} M \cap \operatorname{Ass} N$.

Proof. Recall that $p \in \operatorname{Ass}\operatorname{Hom}_R(M,N)$

$$\begin{aligned}
&\Leftrightarrow \operatorname{Hom}_{R_p}(k(p), \operatorname{Hom}_R(M,N)_p) \neq 0\\
&\Leftrightarrow \operatorname{Hom}_{R_p}(k(p), \operatorname{Hom}_{R_p}(M_p,N_p)) \neq 0\\
&\Leftrightarrow \operatorname{Hom}_{R_p}(k(p)\otimes_{R_p} M_p, N_p) \neq 0\\
&\Leftrightarrow \operatorname{Hom}_{R_p}(k(p)^{\mu(M_p)}, N_p) \neq 0\\
&\Leftrightarrow \operatorname{Hom}_{R_p}(k(p), N_p)^{\mu(M_p)} \neq 0\\
&\Leftrightarrow p \in \operatorname{Ass} N \text{ and } \mu(M_p) \neq 0.
\end{aligned}$$

□

(2) Let (R,m) be a Noetherian local ring, M a finitely generated $R-$module, N any $R-$module. Then $\operatorname{Coass}(M\otimes_R N) = \operatorname{Supp} M \cap \operatorname{Coass} N$.

Proof.

$$\begin{aligned}
\operatorname{Coass}(M\otimes_R N) &= \operatorname{Ass}((M\otimes_R N)^\vee)\\
&= \operatorname{Ass}\operatorname{Hom}_R(M\otimes_R N, E)\\
&= \operatorname{Ass}\operatorname{Hom}_R(M, \operatorname{Hom}_R(N,E))\\
&= \operatorname{Ass}\operatorname{Hom}_R(M, N^\vee)\\
&= \operatorname{Supp} M \cap \operatorname{Ass} N^\vee = \operatorname{Supp} M \cap \operatorname{Coass} N.
\end{aligned}$$

□

Recall. Let R be a local ring of dimension d, $I \subseteq R$, and M an $R-$module. Then $H_I^d(M) = M \otimes_R H_I^d(R)$.

<u>HLVT.</u> If (R,m) is a complete local ring of dimension d, $I \subseteq R$, then $H_I^d(R) \neq 0$ if and only if $\sqrt{I+p} = m$ for some $p \in \operatorname{Spec} R$ such that $\dim R/p = d$.

Lemma 8.6. *Let (R,m) be a complete local ring, $I \subseteq R$, and M a finitely generated $R-$module of dimension n. Then*

$$\operatorname{Coass} H_I^n(M) = \{p \supseteq \operatorname{Ann}_R M \,|\, \dim R/p = n \text{ and } \sqrt{I+p} = m\}.$$

Proof. By the change of rings principle, we may assume $\dim M = \dim R$ and $\operatorname{Ann}_R M = 0$. Notice

$$\operatorname{Coass} H_I^n(M) = \operatorname{Coass}(M \otimes_R H_I^n(R)) = \operatorname{Supp} M \cap \operatorname{Coass} H_I^n(R) = \operatorname{Coass} H_I^n(R)$$

as $\operatorname{Ann}_R M = 0$. We may assume ${}_I^n(R) \neq 0$ as otherwise both sets in the theorem would be empty by HLVT. Let $q \in \operatorname{Coass} H_I^n(R)$. Then $q \in \operatorname{Coass}(R/q \otimes H_I^n(R)) = \operatorname{Supp} R/q \cap H_I^n(R)$. Therefore $R/q \otimes_R H_I^n(R) = H_I^n(R/q) \neq 0$. So $\dim R/q = n$ and $\sqrt{I+q} = m$ by HLVT.

Let $q \in \operatorname{Spec} R$ such that $\dim R/q = n$ and $\sqrt{I+q} = m$. Hence $R/q \otimes_R H_I^n(R) \cong H^n_{(I+q)/q}(R/q) \neq 0$ by HLVT. Let $p \in \operatorname{Coass}(R/q \otimes H_I^n(R)) = \operatorname{Supp} R/q \cap \operatorname{Coass} H_I^n(R)$. So $p \supseteq q$ and $p \in \operatorname{Coass} H_I^n(R)$. But we have shown that if $p \in \operatorname{Coass} H_I^n(R)$ then p is minimal. Thus $p = q$. □

Remark. Let (R,m) be a complete local ring, M, N $R-$modules with M finitely generated and N Artinian. Then $\operatorname{Ext}_R^i(M,N)^\vee \cong \operatorname{Tor}_i^R(M, N^\vee)$.

Proof. If $F.$ is a free resolution of $N^\vee$, then $F.^\vee$ is an injective resolution of $N^{\vee\vee} \cong N$. Then

$$\begin{aligned}
\operatorname{Tor}_i^R(M, N^\vee)^\vee &= H_i(M \otimes_R F.)^\vee \\
&= H^i((M \otimes_R F.)^\vee) \\
&= H^i(\operatorname{Hom}_R(M \otimes_R F., E)) \\
&\cong H^i(\operatorname{Hom}_R(M, F.^\vee)) \\
&\cong \operatorname{Ext}_R^i(M, N).
\end{aligned}$$

□

Definition. *Let (R, m) be a local ring, $I \subseteq R$, and N an $R-$module. N is $I-$**cofinite** if $\operatorname{Supp} N \subseteq V(I)$ and $\operatorname{Ext}_R^i(R/I, N)$ is finitely generated for all i.*

Lemma 8.7. *Let (R, m) be a local ring and $\hat{R}$ the $m-$adic completion of R, $I \subseteq R$ and M an $R-$module. Then $H_I^i(M)$ is $I-$cofinite if and only if $H_{I\hat{R}}^i(M \otimes_R \hat{R})$ is $I\hat{R}-$cofinite.*

Proof. $\operatorname{Ext}_R^i(R/I, H_I^i(M)) \otimes_R \hat{R} \cong \operatorname{Ext}_{\hat{R}}^i(\hat{R}/I\hat{R}, H_{I\hat{R}}^i(M \otimes_R \hat{R}))$. It is enough to show $N \otimes_R \hat{R}$ is finitely generated if and only if N is finitely generated. Of course, this has already been shown. □

Theorem 8.8 (Delfino-Marley, 1997)**.** *Let (R, m) be a Noetherian local ring, $I \subseteq R$, M a finitely generated $R-$module of dimension n. Then $H_i^n(M)$ is $I-$cofinite. In fact, $\operatorname{Ext}_R^i(R/I, H_I^n(M))$ has finite length for all i.*

Proof. By Lemma 8.7, we may assume (R, m) is complete. As $H_I^n(M)$ is Artinian, $H_I^n(M)^\vee$ is finitely generated. Therefore $\operatorname{Coass} H_I^n(M)$ is a finite set, say $\operatorname{Coass} H_I^n(M) = \{p_1, ..., p_k\}$. Then $\operatorname{Supp} H_I^n(M) = V(p_1 \cap \cdots \cap p_k)$. Now $\operatorname{Ext}_R^i(R/I, H_I^n(M))$ has finite length if and only if $\operatorname{Ext}_R^i(R/I, H_I^n(M))^\vee$ has finite length which is if and only if $\operatorname{Tor}_i^R(R/I, H_I^n(M)^\vee)$ has finite length. As $\operatorname{Tor}_i^R(R/I, H_I^n(M)^\vee)$ is a finitely generated $R-$module, it is enough to show its support is $\{m\}$. Now suppose

$$\operatorname{Tor}_i^R(R/I, H_I^n(M)^\vee) \subseteq V(I) \cap \operatorname{Supp} H_I^n(M)^\vee = V(I) \cap V(p_1 \cap \cdots \cap p_k) = V(I + p_1 \cap \cdots \cap p_k) = \{m\}$$

as $\sqrt{I + p_i} = m$ for all i. □

9. Graded Local Cohomology

Let $R = \oplus R_n$ be a $\mathbb{Z}-$graded ring, $x \in R$ a homogeneous element and M a graded $R-$module. Note that M_x is a graded $R-$ and R_x-module, where $\deg \frac{m}{x^n} = \deg m - n \deg x$. Recall an $R-$homomorphism $f : M \to N$ of graded $R-$modules is said to be (homogeneous) of degree 0 if $f(M_n) \subseteq N_n$ for all n. The kernel and image of degree 0 homomorphisms are graded submodules of M and N, respectively.

Now, if M is a graded $R-$module and $\underline{x} = x_1, ..., x_n \in R$ is a sequence of homogeneous elements, then it is easy to see that all the maps in the Čech complex $C^\cdot(\underline{x}; M)$ are degree 0 (In the n=1 case, we have $0 \to M \to M_x \to 0$ defined by $m \mapsto \frac{m}{1}$. Proceed by induction). Therefore, the homology modules $H_{\underline{x}}^i(M)$ are graded $R-$modules. Since every homogenous ideal has a homogeneous set of generators, we get that for all i $H_I^i(M)$ is a graded $R-$module for every homogeneous ideal I of R and graded $R-$module M.

From now on, when we say R is a "graded ring," let us assume R is $\mathbb{N}-$graded. Then R is a Noetherian graded ring if and only if R_0 is Noetherian and $R = R_0[x_1, ..., x_n]$ where $x_1, ..., x_n$ are homogeneous elements in $R_+ = \oplus_{n>0} R_n$. If the x_i can be chosen such that $\deg x_i = 1$ for all i, we say that R is a standard graded ring. Note that the homogeneous maximal ideals of R are of the form $(m_0, R_+)R$ where m_0 is a maximal ideal of R_0. Thus R has a unique homogeneous maximal ideal if and only if R_0 is local. We call such graded rings *local (where *local implies Noetherian).

Proposition 9.1. *Let (R, m) be a *local ring and M a finitely generated graded $R-$module. Then*

(1) *$H_m^i(M)_n = 0$ for all $n >> 0$ and for all i.*

(2) *$H^i_m(M)_n$ is an Artinian R_0-module for all i and for all n.*

Proof. Note that as every element of $H^i_m(M)$ is annihilated by a power of m, $H^i_m(M) \cong H^i_{mR_m}(M_m)$ for all i. In the local case, we showed $H^i_{mR_m}(M_m)$ is Artinian. Thus $H^i_m(M)$ is an Artinian $R-$module. Let $H^i_m(M)_{\geq t} := \oplus_{n\geq t} H^i_m(M)_n$. Then $H^i_m(M)_{\geq t}$ is a graded $R-$module and $H^i_m(M)_{\geq t} \supseteq H^i_m(M)_{\geq t+1} \supseteq \cdots$. By DCC, $H^i_m(M)_{\geq t} = H^i_m(M)_{\geq t+1}$ for all $t >> 0$. Thus $H^i_m(M)_t = 0$ for all $t >> 0$.

For 2, suppose $H^i_m(M)_n = N_0 \supseteq N_1 \supseteq N_2 \supseteq \cdots$ is a descending chain of R_0-submodules of $H^i_m(M)_n$. Then $RN_0 \supseteq RN_1 \supseteq RN_2 \supseteq \cdots$ is a desending chain of $R-$submodules of $H^i_m(M)$. Hence, $RN_t = RN_{t+1}$ for $t >> 0$. Therefore

$$N_t = RN_t \cap H^i_m(M)_n = RN_{t+1} \cap H^i_m(M)_n = N_{t+1}$$

for $t >> 0$. Hence $H^i_m(M)_n$ is an Artinian R_0-module. □

Corollary 9.2. *Suppose in the above proposition that R_0 is Artinian. Then $\lambda_{R_0}(H^i_m(M)_n) < \infty$ for all i, n.*

Proof. An Artinian module over an Artinian ring has finite length. □

Definition. *Let (R, m) be a *local Cohen Macaulay standard graded ring. The $a-$**invariant** of R is defined by $a(R) = \sup\{n | H^d_m(R)_n \neq 0\}$ for $d = \dim R$.*

Example. Let $R = k[x_1, ..., x_d]$ for a field k. Then we have seen

$$H^d_m(R) \cong E_R(R/m) \cong R_{x_1\cdots x_d} \Big/ \sum R_{x_1\cdots\hat{x_i}\cdots x_d} \cong \oplus_{i,j<0} k x_1^{i_1}\cdots x_d^{i_d}.$$

Thus $a(R) = -d$.

Proposition 9.3. *Let (R, m) be a *local Cohen Macaulay standard graded ring. Suppose $x \in R$ is a homogeneous non-zerodivisor on R. Then $a(R/(x)) = a(R) + \deg x$.*

Proof. Consider the exact sequence $0 \to R(-k) \xrightarrow{x} R \to R/(x) \to 0$ (where $k = \deg x$). Then we have

$$0 \to H^{d-1}_m(R/(x)) \to H^d_m(R(-k)) \xrightarrow{x} H^d_m(R) \to 0$$

is exact. These are degree 0 maps and so $0 \to H^{d-1}_m(R/(x))_n \to H^d_m(R)_{n-k} \xrightarrow{x} H^d_m(R)_n \to 0$ is exact. Now $H^{d-1}_m(R/(x))_n \neq 0$ if $n = a(R/(x))$. Therefore $H^d_m(R)_{a(R/(x))-k} \neq 0$ and $a(R) \geq a(R/(x)) - k$.

As $H^{d-1}_m(R/(x))_n = 0$ for $n > a(R/(x))$, $H^d_m(R)_{n-k} \xrightarrow{x} H^d_m(R)_n$ is injective for all $n > a(R/(x))$. But every element in $H^d_m(R)$ is annihilated by a power of x. Thus $H^d_m(R)_n = 0$ for all $n > a(R/(x)) - k$. Thus $a(R) = a(R/(x)) - k$. □

Theorem 9.4. *Let (R, m) be a Cohen Macaulay *local standard graded ring such that R_0 is Artinian. Then $a(R) \geq -\dim R$ with equality if and only if $R \cong R_0[T_1, ..., T_d]$.*

Proof. Assume R/m is infinite (else tensor with $R[T]_{mR[T]}$). Note that as R_0 is Artinian, $m = \sqrt{R_+} = \sqrt{R_1 R}$. Let $n = \mu_{R_0}(R_1)$. Choose minimal generators $x_1, ..., x_n$ for R_1 such that $x_1, ..., x_d$ is an $R-$regular sequence. (We can do this as R is Cohen Macaulay. Choose $x_1 \in R_1 \setminus m_0 R_1 \cup p_1 \cup \cdots \cup p_r$ where $\{p_i\} = \mathrm{Ass}(R)$). Induct on d.

If $d = 0$, $H^0_m(R) = R$ and so $a(R) \geq 0$. Now $a(R) = 0$ if and only if $R = R_0$. Suppose $d > 0$. Then $a(R) = a(R/(x_1)) - 1 \geq -d + 1 - 1 = -d$. Write $R = R_0[T_1, ..., T_n]/I$ where $T_1, ..., T_n$ are indeterminates and $n = \mu_{R_0}(R_1)$. Now $a(R/(\overline{T_1}) = a(R) + 1 = -d + 1$. Thus $R/(\overline{T_1}) = R/(I, T_1) \cong R_0[\overline{T_2}, ..., \overline{T_n}]$. Thus $n - 1 = d - 1$ by induction.

We need to show $I = 0$. We have $I \subseteq (T_1)$. If $I \neq 0$, then there exists $f \notin (T_1)$ such that $fT_1 \in I$ (else $T_1^r \subseteq I$). But this means T_1 is a zerodivisor in R, a contradiction. Thus $I = 0$. □

The $a-$invariant is closely related to the Castelnuovo-Mumford regularity of R.

Definition. *Let (R, m) be a *local standard graded ring of dimension d such that R_0 is Artinian. Define $a_i(R) := \sup\{n | H^i_m(R)_n \neq 0\}$ for $i = 0, ..., d$ (set $a_i(R) = -\infty$ if $H^i_m(R) = 0$). The **Castelnuovo-Mumford regularity** of*

R *is*

$$\operatorname{reg}(R) := \max\{a_i(R) + i | i = 0, ..., d\}.$$

One can prove that $\operatorname{reg}(R) \geq 0$ with equality if and only if $R \cong R_0[T_1, ..., T_d]$.

Definition. *Let R be a *local standard graded ring such that R_0 is Artinian and M a finitely generated graded $R-$module. As each M_n is a finitely generated R_0-module, $\lambda_{R_0}(M_n) < \infty$ for all n. Define the **Hilbert function** of M by $H_M(n) := \lambda_{R_0}(M_n)$.*

Example.

(1) Let $R = k[x_1, ..., x_d]$ for a field k. Then $H_R(n) = \binom{n+d-1}{d-1}$, the number of monomials of degree n in $x_1, ..., x_d$.
(2) Let $R = k[x, y]/(x^3, xy)$. Then $H_R(0) = 1, H_R(1) = 2, H_R(2) = 2, H_R(3) = 1$, and $H_R(n) = 1$ for all $n \geq 3$.

Theorem 9.5. *Let (R, m) be a *local standard graded ring such that R_0 is Artinian and M is a finitely generated $R-$module of dimension n. Then there exists a unique polynomial $P_m(x) \in \mathbb{Q}[x]$ such that $P_m(n) = H_m(n)$ for $n >> 0$. $P_m(x)$ is the **Hilbert polynomial of** M.*

Proof. See Atiyah and Macdonald. □

Definition. *Let $f : \mathbb{Z} \to \mathbb{Z}$ be a function. Define $\Delta : \mathbb{Z} \to \mathbb{Z}$ by $\Delta(f)(n) = f(n) - f(n-1)$.*

Remark. Let $f, g : \mathbb{Z} \to \mathbb{Z}$ be a function. Then $\Delta(f) = \Delta(g)$ if and only if $f - g$ is a constant.

Definition. *Let (R, m) be a *local standard graded ring such that R_0 is Artinian and M is a finitely generated graded $R-$module. Define $\chi_M(n) := \sum_{i=0}^{\infty}(-1)^i\lambda(H_m^i(M)_n)$. Note the sum is finite and $\chi_M(n) = 0$ for $n >> 0$. In fact, $\chi_M(n) = 0$ for $n > \max\{a_0(M), ..., a_d(M)\}$ where $d = \dim M$.*

Lemma 9.6. *Let (R, m) be a *local standard graded ring such that R_0 is Artinian and $0 \to A \to B \to C \to 0$ is a short exact sequence of finitely generated graded $R-$modules with degree 0 maps. Then*

(1) *$H_B(n) = H_A(n) + H_C(n)$ for all n*
(2) *$P_B(x) = P_A(x) + P_C(x)$*
(3) *$\chi_B(n) = \chi_A(n) + \chi_C(n)$ for all n*

Proof. (1) Follows from the exactness of $0 \to A_n \to B_n \to C_n \to 0$ for all n.
(2) We have a long exact sequence with degree 0 maps $\cdots \to H_m^i(A) \to H_m^i(B) \to H_m^i(C) \to \cdots$. So $\cdots \to H_m^i(A)_n \to H_m^i(B)_n \to H_m^i(C)_n \to \cdots$ is exact for all n. Use the additivity of λ.

□

Theorem 9.7. *Let (R, m) be a *local standard graded ring such that R_0 is Artinian and M a finitely generated graded $R-$module. Then $H_M(n) - P_M(n) = \chi_M(n)$ for all n.*

Proof. Let $R = R_0[x_1, ..., x_s]$, where $x_1, ..., x_s \in R_1$. Induct on s. For $s = 0$, $R = R_0$ and $\lambda(M) < \infty$. Thus $M_n = 0$ for $n >> 0$ which implies $P_M(n) = 0$ for all n. So $H_m^0(M) = M$ and $H_m^i(M) = 0$ for all $i > 0$. Therefore $\chi_M(n) = \chi(M_n) = H_M(n)$.

Suppose $s > 0$. Consider the exact sequence $0 \to K \to M(-1) \xrightarrow{x_s} M \to C \to 0$ of graded $R-$modules and degree 0 maps. By the lemma,

$$\Delta(H_M(n) - P_M(n)) = H_M(n)H_M(n-1) - P_M(n) + P_M(n-1) = H_C(n) - P_C(n) - (H_K(n) - P_K(n)).$$

Now $x_rK = 0 = x_rC$, so K and C are R/x_sR-modules. By induction on s,

$$\Delta(H_M(n) - P_M(n)) = \chi_C(n) - \chi_K(n) = \chi_M(n) - \chi_M(n-1) = \Delta(\chi_M(n)).$$

By the remark, $H_M(n) - P_M(n) = \chi_M(n) + C$. But $\chi_M(n) = 0$ for $n >> 0$ and $H_M(n) - P_M(n) = 0$ for $n >> 0$. Thus $C = 0$. □

Corollary 9.8. *Let (R, m) be a Cohen Macaulay *local standard graded ring such that R_0 is Artinian. Then $a(R) = \min\{n \in \mathbb{Z} | P_R(n) \neq H_R(n)\}$.*

Proof. $H_R(n) - P_R(n) = (-1)^d \lambda(H^d_m(R)_n)$. □

Question. Let (R, m) be a local ring, M a finitely generated $R-$module and $I \subseteq R$. When is $H^i_I(M)$ finitely generated?

Certainly it is when $i = 0$. However, not always.

Remark. $H^i_I(M)$ is a finitely generated $R-$module if and only if $H^i_{I\hat{R}}(\hat{M})$ is a finitely generated $\hat{R}-$module.

Proposition 9.9. *Let (R, m) be a local ring and M a finitely generated $R-$module of dimension $n > 0$. Then $H^n_m(M)$ is not finitely generated.*

Proof. If it were, then $H^n_m(M) \otimes R/m \neq 0$. But $H^n_m(M) \otimes R/m) \cong H^n_m(M/mM) = 0$ as $\dim M/mM = 0 < n$. □

Proposition 9.10. *Let R be a Noetherian ring, $I \subseteq R$, and M a finitely generated $R-$module. TFAE*

(1) *$H^i_I(M)$ is finitely generated for all $i \leq t$.*
(2) *$I \subseteq \sqrt{\operatorname{Ann}_R H^i_I(M)}$ for all $i \leq t$, that is, there exists k such that $I^k H^i_I(M) = 0$ for all $i < t$.*

Proof. Note that 1 implies 2 is clear as every element in $H^i_I(M)$ is killed by a power of I. So we need to show 2 implies 1. We will induct on t. The $t = 0$ case is clear so assume $t > 0$. Let $L = H^0_I(M)$ and $N = M/L$. Then $H^0_I(L) = L$ and $H^i_I(L) = 0$ for all $i \geq 1$. Therefore, from the long exact sequence $\cdots \to H^i_I(L) \to H^i_I(M) \to H^i_I(N) \to \cdots$ we get $H^0_I(N) = 0$ and $H^i_I(N) \cong H^i_I(M)$ for all $i \geq 1$. Hence we may assume $\operatorname{depth}_I M > 0$.

Let $x \in I$ such that $x \in I$ is a non-zerodivisor on M. By assumption, there exists k such that $x^k H^i_I(M) = 0$ for all $i \leq t$. As x^k is a non-zerodivisor on M, replace x^k by x. From $0 \to M \xrightarrow{x} M \to M/xM \to 0$, we get $\cdots \xrightarrow{0} H^{t-1}_I(M) \to H^{t-1}_I(M/xM) \to H^t_I(M) \xrightarrow{x} H^t_I(M)$. By induction, $H^i_I(M)$ is finitely generated for all $i \leq t-1$. Also, as $I^k H^i_I(M) = 0$ for all $i \leq t$ and

$$0 \to H^{i-1}_I(M) \to H^{i-1}_I(M/xM) \to H^i_I(M) \to 0$$

is exact for all $i \leq t$, $I^{2k} H^{i-1}_I(M/xM) = 0$ for all $i \leq t$. Therefore $H^{t-1}_I(M/xM)$ is finitely generated, which implies $H^t_I(M)$ is finitely generated. Thus the finite generation of $H^i_I(M)$ is related to the annihilation of $H^i_I(M)$. □

Theorem 9.11 (Faltings, 1978)**.** *Let (R, m) be a local ring which is the homomorphic image of a regular local ring. Let M be a finitely generated $R-$module and $J \subseteq I$ two ideals of R. Set $s = \min_{p \not\supseteq J}\{\operatorname{depth} M_p + \operatorname{ht}(I+p)/p\}$. Then*

(1) *$J \subseteq \sqrt{\operatorname{Ann}_R H^i_I(M)}$ for all $i < s$*
(2) *$J \not\subseteq \sqrt{\operatorname{Ann}_R H^s_I(M)}$.*

Note here we define $\operatorname{depth} M_p = \infty$ if $M_p = 0$ and $\min \emptyset = \infty$. As a corollary, we get the following result.

Theorem 9.12 (Grothendieck, SGAII, 1968)**.** *Let (R, m) be a local ring which is the quotient of a regular local ring. Let M be a finitely generated $R-$module and $I \subseteq R$. Set $s = \min_{p \not\supseteq I}\{\operatorname{depth} M_p + \operatorname{ht}(I+p)/p\}$. Then $H^i_I(M)$ is finitely generated for all $i < s$ and $H^s_I(M)$ is not finitely generated.*

Proof. Set $J = I$ in Falting's Theorem and use the proposition. □

Lemma 9.13. *Let (R, m) be a local ring which is the quotient of a Gorenstein ring. Let M be a finitely generated $R-$modules and $J \subseteq R$ an ideal. Then $J \subseteq \sqrt{\operatorname{Ann}_R H^i_m(M)}$ if and only if for all $p \not\supseteq J$ $H^{i-\dim R/p}_{pR_p}(M_p) = 0$.*

Proof. Let $R = T/I$ where (T, n) is a Gorenstein local ring. Let $K \subseteq T$ such that $K/I = J$. Then by the change of rings principle $J \subseteq \sqrt{\operatorname{Ann}_R H^i_m(M)}$ if and only if $K \subseteq \sqrt{\operatorname{Ann}_T H^i_n(M)}$. Also, if $q \supseteq I$, $q \not\supseteq K$, then $H^{i-\dim T/q}_{qT_q}(M_q) \cong H^{i-\dim R/p}_{pR_p}(M_p)$ where $p = q/I$. If $q \not\supseteq I$, then $M_q = 0$. Hence, we may assume (R, m) is a Gorenstein local ring.

Now $J \subseteq \sqrt{\operatorname{Ann}_R H^i_m(M)}$

$$
\begin{aligned}
&\Leftrightarrow \quad J \subseteq \sqrt{\operatorname{Ann}_R \operatorname{Ext}^{d-i}_R(M,R)^\vee} \\
&\Leftrightarrow \quad J \subseteq \sqrt{\operatorname{Ann}_R \operatorname{Ext}^{d-i}_R(M,R)} \\
&\Leftrightarrow \quad \text{for all } p \not\supseteq J, \operatorname{Ext}^{d-i}_{R_p}(M_p, R_p) = 0 \\
&\Leftrightarrow \quad \text{for all } p \not\supseteq J, H^{\dim R_p - d + i}_{pR_p}(M_p) = 0 \text{ and } d - \dim R_p = \dim R/p.
\end{aligned}
$$

□

Proposition 9.14. *Let (R, m) be a local ring which is the quotient of a Gorenstein ring. Let M be a finitely generated $R-$module and $J \subseteq R$ an ideal. Let $s = \min_{p \not\supseteq J}\{\operatorname{depth} M_p + \dim R/p\}$. Then $J \subseteq \sqrt{\operatorname{Ann}_R H^i_m(M)}$ for all $i < s$ and $J \not\subseteq \sqrt{\operatorname{Ann}_R H^s_m(M)}$.*

Proof. By the lemma, $J \subseteq \sqrt{\operatorname{Ann}_R H^i_m(M)}$ for all $i < t$

$$
\begin{aligned}
&\Leftrightarrow \quad H^{i-\dim R/p}_{pR_p}(M_p) = 0 \text{ for all } p \not\supseteq J, i < t \\
&\Leftrightarrow \quad \text{for all } p \not\supseteq J, t - \dim R/p \leq \operatorname{depth} M_p \\
&\Leftrightarrow \quad t \leq s.
\end{aligned}
$$

□

Lemma 9.15. *Let (R, m) be a Cohen Macaulay local ring, M a finitely generated $R-$module, $I \subseteq R$. Suppose there exists $p \in \operatorname{Spec} R$ such that M_p is free. Then there exists $s \in R \setminus p$ such that $sH^i_I(M) = 0$ for all $i < \operatorname{ht} I$.*

Proof. There exists exact sequences $0 \to C \to F \to T \to 0$ and $0 \to T \to M \to D \to 0$ such that F is a finitely generated free $R-$module and $C_p = D_p = 0$. Choose $s \notin p$ such that $sC = sD = 0$. Then $sH^i_I(C) = sH^i_I(D) = 0$ for all i. Now we have long exact sequences $\cdots \to H^i_I(T) \to H^i_I(M) \to H^i_I(D) \to \cdots$ and $\cdots \to H^i_I(F) \to H^i_I(T) \to H^{i+1}_I(C) \to \cdots$. As R is Cohen Macaulay, $H^i_I(F) = \oplus H^i_I(R) = 0$ for all $i < \operatorname{ht} I$. Thus $sH^i_I(T) = 0$ for all $i < \operatorname{ht} I$. Hence $s^2 H^i_I(M) = 0$ for all $i < \operatorname{ht} I$. □

Proof of part 1 of Falting's Theorem. This proof is due to M. Brodmann in 1983. Set $s(J, I, M) := \min_{p \not\supseteq J}\{\operatorname{depth} M_p + \operatorname{ht}(I + p)/p\}$. We use induction on $\dim R/I$ to prove there exists k such that $J^k H^i_I(M) = 0$ for all $i < s = s(J, I, M)$. The case $\dim R/I = 0$ is taken care of by Proposition 9.14. So assume $\dim R/I > 0$. We make a series of reductions.

Reduction 1. We may assume R is a regular local ring.

Proof. Write $R = T/L$ where T is a regular local ring. Let I', J' be ideals of T such that $I'/L = I$ and $J'/L = J$. Then, as noted in the lemma preceding Proposition 9.14, $s(J', I', M) = s(J, I, M)$ and $H^i_{I'}(M) \cong H^i_I(M)$ for all i. □

Reduction 2. We may assume $s(J, I, M) < \infty$.

Proof. $s(J, I, M) = \infty$ if and only if $M_p = 0$ for all $p \not\supseteq J$, that is, $J \subseteq \sqrt{\operatorname{Ann}_R M}$, which implies there exists k such that $J^k H^i_I(M) = 0$ for all i. □

Reduction 3. We may assume $\operatorname{depth}_J M > 0$.

Proof. Let $N = M/H^0_J(M)$. Note $N \neq 0$ else $J^k M = 0$ for some k, which implies $s(J, I, M) = \infty$. Then, as $H^0_J(M)_p = 0$ for all $p \not\supseteq J$, $M_p \cong N_p$ for all $p \not\supseteq J$. Therefore $s(J, I, M) = s(J, I, N)$. Furthermore, as remarked before, $\operatorname{depth}_J N > 0$. From $0 \to H^0_J(M) \to M \to N \to 0$ we get $\cdots \to H^i_I(H^0_J(M)) \to H^i_I(M) \to H^i_I(N) \to \cdots$. If we know the theorem for N, then $J^k H^i_I(N) = 0$ for all $i < s = s(J, I, M)$. As $J^\ell H^0_J(M) = 0$ for some ℓ, $J^\ell H^i_I(H^0_J(M)) = 0$ for all i. Therefore $J^{\ell+k} H^i_I(M) = 0$ for all $i < s$. □

Reduction 4. We may assume $J \supseteq \operatorname{Ann}_R M$.

Proof. By the change of rings principle, $H^i_I(M) \cong H^i_{IR/\operatorname{Ann}_R M}(M) \cong H^i_{I+\operatorname{Ann}_R M}(M)$ for all i. Also, as $\operatorname{Ann}_R M \subseteq \operatorname{Ann}_R H^i_I(M)$ for all i, we have $J \subseteq \sqrt{\operatorname{Ann}_R H^i_I(M)}$ if and only if $J + \operatorname{Ann}_R M \subseteq \sqrt{\operatorname{Ann}_R H^i_I(M)}$. Finally, if $p \not\supseteq \operatorname{Ann}_R M$ then depth $M_p = \infty$. Hence $s(J + \operatorname{Ann}_R M, I + \operatorname{Ann}_R M, M) = s(J, I, M)$. $\square$

Claim 1. $s(J, I, M) \leq \operatorname{ht} I$. Furthermore, if $s(J, I, M) = \operatorname{ht} I$ then $\operatorname{Ann}_R M = 0$.

Proof. Let q be a prime minimal over I such that $\operatorname{ht} q = \operatorname{ht} I = h$. As $I \supseteq J \supseteq \operatorname{Ann}_R M$, q contains a prime p which is minimal over $\operatorname{Ann}_R M$. Then $p \in \operatorname{Ass}_R M$ and so $p \not\supseteq J$ as $\operatorname{depth}_J M > 0$. Therefore, $s(J, I, M) \leq \operatorname{depth} M_p + \operatorname{ht}(I + p)/p \leq \operatorname{ht} q/p \leq h$.

If we have equality, then (as R is a domain), $p = 0$. Therefore $\operatorname{Ann}_R M = 0$. $\square$

<u>Case 1.</u> Assume $s := s(J, I, M) = \operatorname{ht} I =: h$. By the claim, $\operatorname{Ann}_R M = 0$. Let $U = \{p \in \operatorname{Spec} R | M_p \text{ is free}\}$. Then $U \neq \emptyset$ as $M_{(0)}$ is free and U is open. Let $U = \operatorname{Spec} R - V(L)$, for $L \subseteq R$. Let $\gamma := \{p \in \operatorname{Min} R/L | p \not\supseteq J\}$.

<u>Case 1a.</u> $\Gamma = \emptyset$. Then $p \not\supseteq J$, which implies $p \not\supseteq L$ and M_p is free. By Lemma 9.15, for all $p \not\supseteq J$ there exists $S_p \notin p$ such that $s_p H^i_I(M) = 0$ for all $i < h = s$. Let $A = (\{s_p\}_{p \not\supseteq J})R$. Then $AH^i_I(M) = 0$ for all $i < s$. Furthermore, $J \subseteq \sqrt{A}$ for if $q \in \operatorname{Spec} R$ with $q \supseteq A$ then $q \supseteq J$ (else $s_q \in A, s_q \notin q$). Therefore there exists k such that $J^k H^i_I(M) = 0$ for all $i < s$.

<u>Case 1b.</u> $\Gamma \neq \emptyset$. Let $\Gamma = \{p_1, ..., p_s\}$ and let $\{q_1, ..., q_t\}$ be the minimal primes of height h.

Claim 2. $\cap_{i=1}^s p_i \not\subseteq \cup_{i=1}^t q_i$.

Proof. Suppose not. Then $p_i \subseteq q_j$ for some j. Then M_{p_i} is not free as $p_i \notin U$. By Auslander-Buchsbaum, this means depth $M_{p_i} M \dim R_{p_i}$. Therefore as $p_i \not\supseteq J$

$$s \leq \operatorname{depth} M_{p_i} + \operatorname{ht}(I + p_i)/p_i M \dim R_{p_i} + \operatorname{ht} q_j/p_i = \operatorname{ht} q_j = h,$$

a contradiction. $\square$

So chose $x \in \cap_{i=1}^s p_i \setminus \cup_{i=1}^t q_i$. Note that $\dim R/(I, x) < \dim R/I$ as $x \notin \cup_{i=1}^t q_i$ and if $p \not\supseteq J$ and $x \notin p$, then M_p is free (else, $p \supseteq L$ implies $p \supseteq p_i$ for some i, a contradiction as $x \in p_i$).

Claim 3. $J \subseteq \sqrt{\operatorname{Ann}_R H^i_{I_x}(M_x)}$ for all $i < s = h$.

Proof. It is enough to show $J_x \subseteq \sqrt{\operatorname{Ann}_{R_x} H^i_{I_x}(M_x)}$ for all $i < h$. Now for all $p_x \in \operatorname{Spec}(R_x)$, $p_x \not\supseteq J_x$ and so $(M_x)_{p_x} \cong M_p$ is free. Thus by the same argument in Case 1a there exists k such that $J_x^k H^i_{I_x}(M_x) = 0$ for all $i < \operatorname{ht}(I_x) = h$. $\square$

Claim 4. $J \subseteq \sqrt{\operatorname{Ann}_R H^i_{(I,x)}(M)}$ for all $i < s$.

Proof. Note that as $\operatorname{ht}(((I, x) + p)/p) \geq \operatorname{ht}((I + p)/p)$ for all p, $s' = s(J, (I, x), M) \geq s$. As $\dim R/(I, x) < \dim R/I$, we have the claim by induction. $\square$

Now we have the long exact sequence $\cdots \to H^i_{(I,x)}(M) \to H^i_I(M) \to H^i_{I_x}(M_x) \to \cdots$. So case 1 follows from claims 3 and 4.

<u>Case 2.</u> $s < h$. We use induction on $s - h \geq 0$ (the case $s - h = 0$ is case 1. Let F be a finitely generated $R-$module such that $0 \to K \to F \to M \to 0$ is exact.

Claim 5. $s' := s(J, I, K) > s$.

Proof. Let $p \in \operatorname{Spec} R$ with $p \not\supseteq J$. If M_p is free, then K_p is free. Thus depth $K_p + \operatorname{ht}((I + p)/p) = \dim R_p + \operatorname{ht}((I + p)/p) = \operatorname{ht}(I + p) \geq \operatorname{ht} I > s$. If M_p is not free, then $\operatorname{pd} K_p = \operatorname{pd} M_p - 1$. By Auslander Buchsbaum, depth $K_p = \operatorname{depth} M_p + 1$. Thus depth $K_p + \operatorname{ht}((I + p)/p) > \operatorname{depth} M_p + \operatorname{ht}((I + p)/p) \geq s$. $\square$

Thus $h - s' < h - s$ (note that $\operatorname{depth}_J K > 0$ and $\operatorname{Ann}_R K = 0$ as $K \subseteq F$ and R is a domain and so claim 1 still holds). By induction, $J \subseteq \sqrt{\operatorname{Ann}_R H_I^i(K)}$ for all $i < s'$ (hence for $i+1 < s$). As R is a regular local ring, $H_I^i(F) = 0$ for all $i < h(> s)$. From the long exact sequence $\cdots \to H_I^i(F) \to H_I^i(M) \to H_I^{i+1}(K)$, we get $J \subseteq \sqrt{\operatorname{Ann}_R H_I^i(M)}$ for all $i < s$. □

Proof of part 2 of Falting's Theorem. Let $s(J, I, M) = \min_{p \not\supseteq J}\{\operatorname{depth} M_p + \operatorname{ht}((I+p)/p)\}$. We will show that if $s = s(J, I, M) < \infty$ then $J \not\subset \sqrt{\operatorname{Ann}_R H_I^i(M)}$ for some $i \leq s$. As in the proof of part 1, we may replace M by $M/H_J^0(M)$ and assume $\operatorname{depth}_J M > 0$. Induct on s. Note that if $p \not\supseteq J$ then $\operatorname{ht}((I+p)/p) \geq 1$. Thus $s \geq 1$. So first suppose $s = 1$. Choose $p \not\supseteq J$ such that $1 = \operatorname{depth} M_p + \operatorname{ht}((I+p)/p)$. Then $\operatorname{depth} M_p = 0$ and $\operatorname{ht}((I+p)/p) = 1$. Then $p \in \operatorname{Ass}_R M$ and so there exists an exact sequence $0 \to R/p \to M \to N \to 0$. Therefore, $0 \to H_I^0(N) \to H_I^1(R/p) \to H_I^1(M)$ is exact.

Suppose $J \subset \sqrt{\operatorname{Ann}_R H_I^1(M)}$. As $H_I^0(N)$ is finitely generated, $J \subseteq I \subseteq \sqrt{\operatorname{Ann}_R H_I^0(N)}$. Thus $H \subseteq \sqrt{\operatorname{Ann}_R H_I^1(R/p)}$. As $\operatorname{ht}((I+p)/p) = 1$, choose $q \supseteq I+p$ such that $\operatorname{ht} q/p = 1$. Then $J_q \subseteq \sqrt{\operatorname{Ann}_R H_{I_q}^1(R_q/p_q)}$. Let $A = R_q/p_q$ with maximal ideal n. Then A is a one-dimensional local domain. As $p \not\supseteq J$, $\sqrt{J_q A} = \sqrt{I_q A} = n$. Hence $n = \sqrt{\operatorname{Ann}_R H_n^1(A)}$ which implies $H_n^1(A)$ is finitely generated, a contradiction.

Now suppose $s > 1$. Choose $p \not\subseteq J$ such that $s = \operatorname{depth} M_p + \operatorname{ht}((I+p)/p)$. Let q be a prime which contains $I+p$ such that $\operatorname{ht}(q/p) = \operatorname{ht}((I+p)/p)$. Let $y \in J \setminus p$ and consider the set $\Gamma = \{Q \in \operatorname{Spec} R | p \subseteq Q \subseteq q, y \notin Q\}$. As $p \in \Gamma$, we see $\Gamma \neq \emptyset$. Choose $Q \in \Gamma$ maximal. Clearly $Q \not\supseteq J$.

Claim 1. $\operatorname{ht} q/Q = 1$.

Proof. Clearly $q \subsetneq Q$ as $y \in J \subseteq I \subseteq q$. Suppose $\operatorname{ht}(q/Q) > 1$. By prime avoidance and Krull's principle ideal theorem, there exists $Q_1 \subseteq q$ such that $y \notin Q_1$ and $\operatorname{ht}(Q_1/Q) > 0$. But then $Q_1 \in \Gamma$, contradiction to maximality. □

Claim 2. $s = \operatorname{depth} M_q + \operatorname{ht}((I+Q)/Q)$.

Proof. By definition of s we have $s = \operatorname{depth} M_p + \operatorname{ht}((I+p)/p) \leq \operatorname{depth} M_Q + \operatorname{ht}((I+Q)/Q)$. Also,

$$\begin{aligned}
\operatorname{depth} M_Q + \operatorname{ht}((I+Q)/Q) &\leq \operatorname{depth} M_Q + \operatorname{ht}(q/Q) \\
&\leq \operatorname{depth} M_p + \operatorname{ht}(Q/p) + \operatorname{ht}(q/Q)(*) \\
&\leq \operatorname{depth} M_p + \operatorname{ht}(q/p) \\
&= \operatorname{depth} M_p + \operatorname{ht}((I+p)/p).
\end{aligned}$$

(*) To see this inequality, we need to show that if (R, m) is local and M a finitely generated $R-$module and $p \in \operatorname{Spec} R$ then $\operatorname{depth} M \leq \operatorname{depth} M_p + \dim R/p$. But this follows from Ischebeck's Theorem (Mats, Theorem 17.1). □

By Claim 1, q is minimal over $I+Q$ and $\operatorname{ht}(q/Q) = 1$. Replace Q by P (so we may assume $\operatorname{ht}((I+p)/p) = 1$). It is enough to show $J_q \not\subset \sqrt{\operatorname{Ann}^i_{I_q}(M_q)}$ for some $i \leq s$. Therefore, localize at q and assume $q = m$. Hence $s = \operatorname{depth} M_p + \dim R/p = \operatorname{depth} M_p + 1$.

Claim 3. p contains a non-zerodivisor.

Proof. If not, p is contained in an associated prime of M. As $\dim R/p = 1$ and $\operatorname{depth}_J M > 0$, $p \in \operatorname{Ass}_R M$. Then $\operatorname{depth} M_p = 0$ and $s = 1$, a contradiction as $s > 1$. □

Now let $x \in p$ be a non-zerodivisor on M. Then $0 \to M \xrightarrow{M} M/xM \to 0$ is exact. Note that $s' = s(J, I, M/xM) \leq s - 1$ as $\operatorname{depth}(M/xM)_p = \operatorname{depth} M_p - 1$. Therefore, for some $i \leq s-1$, $J \not\subset \sqrt{\operatorname{Ann}_R H_I^i(M/xM)}$. From $\cdots \to H_I^i(M) \to H_I^i(M/xM) \to H_I^{i+1}(M) \to \cdots$ we see that $J \not\subseteq \sqrt{\operatorname{Ann}_R H_I^i(M)}$ for some $i \leq s$. □