LEVEL AND GORENSTEIN PROJECTIVE DIMENSION

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Abstract. We investigate the relationship between the level of a bounded complex over a commutative ring with respect to the class of Gorenstein projective modules and other invariants of the complex or ring, such as projective dimension, Gorenstein projective dimension, and Krull dimension. The results build upon work done by J. D. Christensen [7], H. Altmann et al. [1], and Avramov et al. [4] for levels with respect to the class of finitely generated projective modules.

1. Introduction

The concept of *level* in a triangulated category, first defined by Avramov, Buchweitz, Iyengar, and Miller [4], is a measure of how many mapping cones (equivalently, extensions) are needed to build an object from a collection of other objects, up to suspensions, finite sums, and retractions. This concept has its origins in the works of Beilinson, Bernstein, and Deligne [5], J. D. Christensen [7], Bondal and Van den Bergh [6], Rouquier [15], and others. In particular, the concept of level is implicit in Rouquier's definition of dimension of a triangulated category.

In the case of the bounded derived category of a commutative Noetherian local ring, levels have been used to establish, among other things, a lower bound on the sum of the Loewy lengths of the homology modules of (non-acyclic) perfect complexes ([4, Theorem 10.1]). In this context, it is interesting to compare the level of an object with other more familiar homological invariants. For instance, the level of a finitely generated module (considered as a complex concentrated in degree zero) with respect to the ring is one more than the projective dimension of the module ([7]; see also [1, Cor. 2.2]). On the other hand, since the level of an object and its suspension are the same, uniform bounds on levels may exist in situations where there are no such bounds for homological dimensions. For example, a local ring is regular if and only if the level with respect to the ring of any bounded complex with finite homology is at most one more than the dimension of the ring [4, Theorem 5.5], whereas the projective dimensions of such complexes over a regular local ring may be arbitrarily large.

In this paper, we study the levels of complexes with respect to the class G of Gorenstein projective modules in the bounded derived category of a commutative ring R. It is straightforward to prove that if M is a bounded complex (by which we always mean homologically bounded) which is not acyclic then

$$\operatorname{level}_R^{\mathsf{G}} M \leqslant \operatorname{Gpd}_R M - \inf M + 1,$$

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where $\operatorname{Gpd}_R M$ is the Gorenstein projective dimension of M and $\inf M = \inf\{n \mid \mathrm{H}_n(M) \neq 0\}$. (Here we use $\operatorname{level}_R^{\mathsf{G}} M$ in place of $\operatorname{level}_{\mathsf{D}(R)}^{\mathsf{G}} M$; see Notation 2.6.) Typically, lower bounds for the level of an object are much harder to obtain. One of our main results is the following:

Theorem 1.1. *Let M be a bounded below complex which is not acyclic. Then*

$$\operatorname{level}_R^{\mathsf{G}} M \geqslant \operatorname{Gpd}_R M - \sup M + 1.$$

As an immediate consequence, we obtain the following generalization of [7, Proposition 4.5] and [1, Cor 2.2]:

Corollary 1.2. *Let M be a nonzero R-module of finite Gorenstein projective dimension. Then*

$$\operatorname{level}_R^{\mathsf{G}} M = \operatorname{Gpd}_R M + 1.$$

As with the proofs of [7, Proposition 4.5] and [1, Cor 2.2], our proof of Theorem 1.1 relies critically on an application of the "Ghost Lemma" (cf. [12, Theorem 3]). However, the argument here is significantly more complicated, as maps between (hard) truncations of Gorenstein projective resolutions are not necessarily G-ghost. As a consequence, we are able to prove the following characterization of Gorenstein local rings:

Corollary 1.3. *Let R be a local Noetherian ring with residue field k. The following conditions are equivalent:*

(a) *R is Gorenstein;*
(b) $\operatorname{level}_R^{\mathsf{G}} k = \dim R + 1$;
(c) $\operatorname{level}_R^{\mathsf{G}} M \leqslant \dim R + \sup M - \inf M + 1$ *for all non-acyclic bounded below complexes M.*

Finally, as mentioned above, it is known that a commutative Noetherian local ring R is regular if and only if $\operatorname{level}_R^R M \leqslant \dim R + 1$ for every bounded complex M over R with finitely generated homology ([4, Theorem 5.5]). We show that a direct analogue of this result for Gorenstein rings and $\operatorname{level}_R^{\mathsf{G}} M$ in place of $\operatorname{level}_R^R M$ does not hold (Example 3.10). However, we are able to establish a global bound on G-level over arbitrary Gorenstein local rings:

Theorem 1.4. *Let R be a Gorenstein local ring of dimension d and M a complex in $\mathsf{D}_b(R)$. Then*

$$\operatorname{level}_R^{\mathsf{G}} M \leqslant 2d + 2.$$

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2. Preliminaries

Throughout this paper R will denote a commutative ring with identity.

2.1. **Complexes and derived categories.** We will work with complexes over R, which we grade homologically:

$$M := \cdots \to M_{n+1} \xrightarrow{\partial_{n+1}} M_n \xrightarrow{\partial_n} M_{n-1} \to \cdots$$

For each n we let $\mathrm{Z}_n(X) := \ker \partial_n$, $\mathrm{B}_n(X) := \operatorname{im} \partial_{n+1}$, $\mathrm{C}_n(X) := \operatorname{coker} \partial_{n+1}$, and $\mathrm{H}_n(M) = \mathrm{Z}_n(M)/\mathrm{B}_n(M)$. We also set $\sup M := \sup\{n \mid \mathrm{H}_n(M) \neq 0\}$ and $\inf M := \inf\{n \mid \mathrm{H}_n(M) \neq 0\}$.

We use the notation $\mathsf{D}(R)$ to denote the derived category of R. Similarly, we let $\mathsf{D}_+(R)$ (respectively, $\mathsf{D}_b(R)$) denote the full subcategory of $\mathsf{D}(R)$ consisting of all R-complexes M such that $\inf M > -\infty$, (respectively, $\sup M < \infty$ and $\inf M > -\infty$). We let $\mathsf{D}^f(R)$ denote the full subcategory of $\mathsf{D}(R)$ consisting of all complexes whose homology is finitely generated in each degree. The subcategories $\mathsf{D}^f_+(R)$ and $\mathsf{D}^f_b(R)$ are defined similarly. We use the symbol $\simeq$ to denote isomorphism in derived categories. For any R-complex M we let $M^{\#}$ denote the R-complex which is equal to M as a graded R-module but whose differentials are all zero. We refer the reader to [2] for any unexplained terminology or notation regarding complexes.

2.2. **Gorenstein projective dimension.** In this subsection we summarize the basic properties of Gorenstein projective modules and Gorenstein projective dimension for complexes.

Definition 2.1. A complex P of projective R-modules is called *totally acyclic* if it is acyclic and $\operatorname{Hom}_R(P, L)$ is also acyclic for every projective R-module L. An R-module G is called *Gorenstein projective* if G is isomorphic to the cokernel of some differential of a totally acyclic complex.

Definition 2.2. ([9, 1.7]) Let M be a complex in $\mathsf{D}_+(R)$. Let A be the class of all R-complexes X such that $X^{\#}$ is bounded below and X_i is Gorenstein projective for all i. The *Gorenstein projective dimension* of M is defined by

$$\operatorname{Gpd}_R M := \inf\{\sup X^{\#} \mid X \in \mathsf{A} \text{ and } X \simeq M \text{ in } \mathsf{D}_+(R)\}.$$

Proposition 2.3. ([9, Theorem 3.1]) *Let M be a complex in $\mathsf{D}_+(R)$. The following are equivalent:*

(a) $\operatorname{Gpd}_R M \leqslant n$.
(b) $n \geqslant \sup M$ *and for any (equivalently, some) R-complex $X \in \mathsf{A}$ with $X \simeq M$, the module $\mathrm{C}_n(X)$ is Gorenstein projective.*

As the terminology suggests, Gorenstein projective dimension can be used to characterize Gorenstein rings:

Theorem 2.4. ([8, 4.4.5 and 1.4.9]) *Let R be a Noetherian local ring with residue field k. The following are equivalent:*

(a) R is Gorenstein;
(b) $\operatorname{Gpd}_R k < \infty$*;*
(c) $\operatorname{Gpd}_R M \leqslant \sup M + \dim R$ *for all nonzero complexes M in $\mathsf{D}_+(R)$.*

2.3. **Levels in triangulated categories.** We adopt the notation and terminology of Section 2 of [4] regarding levels in a triangulated category.

Let T be a triangulated category. A subcategory of T is called *strict* if it is closed under isomorphisms in T. A full triangulated subcategory of T is called *thick* if it is strict and closed under direct summands. It is readily seen that the intersection of thick subcategories is again thick. Let C be a nonempty collection of objects of T. The *thick closure* of C, denoted $\operatorname{thick}_{\mathsf{T}}(\mathsf{C})$, is defined to be the intersection of all thick subcategories of T containing C. We let $\operatorname{add}(\mathsf{C})$ (respectively, $\operatorname{add}^{\Sigma}(\mathsf{C})$) denote the the intersection of all strict and full subcategories of T which contain C and are closed under finite sums (respectively, closed under finite sums and suspensions). We let $\operatorname{smd}(\mathsf{C})$ denote the intersection of all strict and full subcategories of T which contain C and are closed under direct summands. Let A and B be strict and full subcategories of T. We define $\mathsf{A}\star\mathsf{B}$ to be the full subcategory of T whose objects consist of all objects M of T such that there exists an exact triangle $L \to M \to N \to \Sigma L$ where $L \in \mathsf{A}$ and $N \in \mathsf{B}$. Evidently, $\mathsf{A}\star\mathsf{B}$ is also strict.

For a collection of objects C of T and a nonnegative integer n, we define the *nth thickening* of C in T to be the full subcategory of T whose objects are defined as follows:

$$\begin{aligned}
&\operatorname{thick}^0_{\mathsf{T}}(\mathsf{C}) = \{0\};\\
&\operatorname{thick}^1_{\mathsf{T}}(\mathsf{C}) = \operatorname{smd}(\operatorname{add}^{\Sigma}(\mathsf{C}));\\
&\operatorname{thick}^n_{\mathsf{T}}(\mathsf{C}) = \operatorname{smd}(\operatorname{thick}^{n-1}_{\mathsf{T}}(\mathsf{C}) \star \operatorname{thick}^1_{\mathsf{T}}(\mathsf{C})) \text{ for } n \geqslant 2.
\end{aligned}$$

It is straightforward to show that $\operatorname{thick}^n_{\mathsf{T}}(\mathsf{C}) \subseteq \operatorname{thick}^{n+1}_{\mathsf{T}}(\mathsf{C})$ for all $n \geqslant 0$ and that

$$\operatorname{thick}_{\mathsf{T}}(\mathsf{C}) = \bigcup_{n\geqslant 0} \operatorname{thick}^n_{\mathsf{T}}(\mathsf{C}).$$

For an object M of T we define the C-*level* of M in T by

$$\operatorname{level}^{\mathsf{C}}_{\mathsf{T}} M := \inf\{n \geqslant 0 \mid M \in \operatorname{thick}^n_{\mathsf{T}}(\mathsf{C})\}.$$

Note that $\operatorname{level}^{\mathsf{C}}_{\mathsf{T}}(M) < \infty$ if and only if $M \in \operatorname{thick}_{\mathsf{T}}(\mathsf{C})$.

We list a few basic properties regarding levels:

Proposition 2.5. (cf. [4, Lemma 2.4]) *Let T be a triangulated category and C a nonempty collection of objects of T. Let L, M and N be objects of T.*

(1) $\operatorname{level}^{\mathsf{C}}_{\mathsf{T}} M = \operatorname{level}^{\mathsf{C}}_{\mathsf{T}} N$ *if M is isomorphic to N.*
(2) $\operatorname{level}^{\mathsf{C}}_{\mathsf{T}} M = \operatorname{level}^{\mathsf{C}}_{\mathsf{T}}(\Sigma^s M)$ *for all integers s.*
(3) $\operatorname{level}^{\mathsf{C}}_{\mathsf{T}} M = \operatorname{level}^{\mathsf{D}}_{\mathsf{T}} M$ *where* $\mathsf{D} = \operatorname{smd}(\operatorname{add}(\mathsf{C}))$.
(4) $\operatorname{level}^{\mathsf{C}}_{\mathsf{T}} M = \operatorname{level}^{\mathsf{C}}_{\mathsf{S}} M$ *for any thick subcategory S of T containing C.*
(5) $\operatorname{level}^{\mathsf{C}}_{\mathsf{T}} V \leqslant \operatorname{level}^{\mathsf{C}}_{\mathsf{T}} U + \operatorname{level}^{\mathsf{C}}_{\mathsf{T}} W$ *whenever $U \to V \to W \to \Sigma U$ is an exact triangle in T.*
(6) $\operatorname{level}^{\mathsf{C}}_{\mathsf{T}}(M \oplus N) = \max\{\operatorname{level}^{\mathsf{C}}_{\mathsf{T}} M, \operatorname{level}^{\mathsf{C}}_{\mathsf{T}} N\}$.

Notation 2.6. Let C be a collection of objects in $\mathsf{D}(R)$ and M an R-complex. We let $\operatorname{level}^{\mathsf{C}}_R M$ denote $\operatorname{level}^{\mathsf{C}}_{\mathsf{D}(R)} M$. Then by part (4) of Proposition 2.5, $\operatorname{level}^{\mathsf{C}}_R M = \operatorname{level}^{\mathsf{C}}_{\mathsf{T}} M$ for any thick subcategory T of $\mathsf{D}(R)$ containing C. In the case C consists of a single object, say $\mathsf{C} = \{A\}$, we denote $\operatorname{level}^{\mathsf{C}}_R M$ by $\operatorname{level}^{A}_R M$. Note that $\operatorname{level}^{R}_R M = \operatorname{level}^{\widetilde{\mathsf{P}}}_R M$, where $\widetilde{\mathsf{P}}$ is the class of finitely generated projective modules, since $\widetilde{\mathsf{P}} = \operatorname{smd}(\operatorname{add} R)$ and by Proposition 2.5(3). We'll reserve the symbol P to denote the class of all projective R-modules. Analogously, we let G (respectively, $\widetilde{\mathsf{G}}$) denote the class of all Gorenstein projective modules (respectively, finitely generated Gorenstein projective modules).

The next result follows readily from Proposition 2.5:

Corollary 2.7. *Let C be a collection of objects of $\mathsf{D}(R)$ and M an R-complex. Then*

$$\operatorname{level}_R^{\mathsf{C}} M \leqslant \inf \left\{ \sum_{i\in\mathbb{Z}} \operatorname{level}_R^{\mathsf{C}} L_i \mid L \simeq M \ \textit{in}\ \mathsf{D}(R) \right\}.$$

In particular, if $M \simeq L$ *in* $\mathsf{D}(R)$ *and* $L_i \in \operatorname{thick}^1_{\mathsf{D}(R)}(C)$ *for all* i*, then*

$$\operatorname{level}_R^{\mathsf{C}} M \leqslant \sup L^{\#} - \inf L^{\#} + 1.$$

As a consequence, we have:

Corollary 2.8. *Let* M *be a nonzero complex in* $\mathsf{D}_+(R)$*. Then*

$$\operatorname{level}_R^{\mathsf{P}} M \leqslant \operatorname{pd}_R M - \inf M + 1.$$

Moreover, if R *is Noetherian and* M *is in* $\mathsf{D}_+^f(R)$ *then*

$$\operatorname{level}_R^{\widetilde{\mathsf{P}}} M \leqslant \operatorname{pd}_R M - \inf M + 1.$$

Identical inequalities hold with G *and* $\widetilde{\mathsf{G}}$ *in place of* P *and* $\widetilde{\mathsf{P}}$*, respectively, and* $\operatorname{Gpd}_R M$ *in place of* $\operatorname{pd}_R M$*.*

2.4. **Ghost maps and the Ghost Lemma.**

Definition 2.9. Let T be a thick subcategory of $\mathsf{D}(R)$ and C a collection of objects from T. A morphism $f : M \to N$ in T is called C-*ghost* (respectively, C-*coghost*) if the induced maps $\operatorname{Ext}_R^n(A, M) \to \operatorname{Ext}_R^n(A, N)$ (respectively, $\operatorname{Ext}_R^n(N, A) \to \operatorname{Ext}_R^n(M, A)$) are zero for all n and all complexes A in C.

Remark 2.10. If R is Noetherian and M and N are in $\mathsf{D}^f(R)$ then f is P-coghost if and only if f is $\widetilde{\mathsf{P}}$-coghost, since the functors $\operatorname{Ext}_R^n(M, -)$ and $\operatorname{Ext}_R^n(N, -)$ commute with (arbitrary) direct sums.

A key tool for obtaining lower bounds on levels is the following result, known as the "Ghost Lemma". It was first proved by G. Kelly in 1965 [12, Theorem 3]. (See also [15, Lemma 4.11].) There is a version for both ghost maps and coghost maps:

Theorem 2.11. *Let* T *be a triangulated category,* C *a collection of objects of* T*, and* $f_i : X_i \to X_{i+1}$ *for* $0 \leqslant i \leqslant n-1$ *a sequence of maps in* T *such that* $f_{n-1}f_{n-2}\cdots f_0$ *is a nonzero morphism in* T*.*

(a) (Ghost Lemma) *If each* f_i *is* C*-ghost then* $\operatorname{level}_{\mathsf{T}}^{\mathsf{C}} X_0 \geqslant n+1$*.*

(b) (Coghost Lemma) *If each* f_i *is* C*-coghost then* $\operatorname{level}_{\mathsf{T}}^{\mathsf{C}} X_n \geqslant n+1$*.*

There is an important converse to the Coghost Lemma in the case $\mathsf{T} = \mathsf{D}_b^f(R)$ proved by Oppermann and Šťovíček [14, Theorem 24]:

Theorem 2.12. *Suppose* R *is Noetherian,* M *and* C *are objects in* $\mathsf{D}_b^f(R)$*, and that* $\operatorname{level}_R^{C} M \geqslant n+1$ *for some* $n \geqslant 1$*. Then there exist* C*-coghost maps* $f_i : M_i \to M_{i+1}$ *for* $0 \leqslant i \leqslant n-1$ *in* $D_b^f(R)$ *with* $M_n = M$ *and* $f_{n-1}f_{n-2}\cdots f_0$ *a nonzero morphism.*

We note that a converse of the Ghost Lemma for $\mathsf{D}_b^f(R)$ has been proved by J. Letz in the case R is a quotient of a Gorenstein ring of finite dimension [13, 2.13]. However, it is unknown whether such a result holds for all commutative Noetherian rings.

3. Main Results

We begin with a couple of technical results:

Lemma 3.1. *Consider a diagram of R-modules and R-linear maps with exact rows and such that the squares commute:*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \xrightarrow{f} & B & \xrightarrow{g} & C & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \alpha} & \swarrow{\scriptstyle \exists\phi} & \downarrow{\scriptstyle \beta} & & \downarrow{\scriptstyle \gamma} & & \\ 0 & \longrightarrow & D & \xrightarrow{i} & E & \xrightarrow{j} & F & \longrightarrow & 0. \end{array}$$

Suppose that

(1) there exists an R-linear map $\phi : B \to D$ such that $\alpha = \phi f$, and
(2) the induced map $\operatorname{Ext}^1_R(F, D) \to \operatorname{Ext}^1_R(C, D)$ is injective.

Then the bottom row splits.

Proof. Applying $\operatorname{Hom}_R(-, D)$ we get a commutative diagram with exact rows:

$$\begin{array}{ccccc} \operatorname{Hom}_R(E, D) & \xrightarrow{i^*} & \operatorname{Hom}_R(D, D) & \xrightarrow{\delta} & \operatorname{Ext}^1_R(F, D) \\ \downarrow & & \downarrow{\scriptstyle \alpha^*} & & \downarrow \\ \operatorname{Hom}_R(B, D) & \xrightarrow{f^*} & \operatorname{Hom}_R(A, D) & \xrightarrow{\epsilon} & \operatorname{Ext}^1_R(C, D) \end{array}$$

Note that $\alpha^*(1_D) = \alpha = \phi f = f^*(\phi) \in \operatorname{im} f^* = \ker \epsilon$. Thus $1_D \in \ker \delta = \operatorname{im} i^*$, by the assumed injectivity of the right-most vertical map. That is, $1_D = \sigma i$ for some $\sigma : E \to D$. □

Lemma 3.2. *Let $f : P \to Q$ be a quasi-isomorphism of R-complexes of Gorenstein projective modules such that $P^\#$ and $Q^\#$ are bounded below. Then for any R-module M of finite projective dimension and for all integers $i \geqslant 1$ and all v, we have the following isomorphisms induced by f:*

(a) $\operatorname{Ext}^i_R(\mathrm{C}_v(Q), M) \cong \operatorname{Ext}^i_R(\mathrm{C}_v(P), M)$;
(b) $\operatorname{Ext}^i_R(\mathrm{B}_v(Q), M) \cong \operatorname{Ext}^i_R(\mathrm{B}_v(P), M)$.

Proof. Let $n = \min\{\inf P^\#, \inf Q^\#\}$. Both isomorphisms clearly hold for all i and $v < n$. Let $j \geqslant n$ and assume the isomorphisms hold for all $i \geqslant 1$ and all $v < j$. We have the following commutative diagram where the vertical arrows are induced by f:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathrm{H}_j(P) & \longrightarrow & \mathrm{C}_j(P) & \longrightarrow & \mathrm{B}_{j-1}(P) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathrm{H}_j(Q) & \longrightarrow & \mathrm{C}_j(Q) & \longrightarrow & \mathrm{B}_{j-1}(Q) & \longrightarrow & 0. \end{array}$$

Since f is a quasi-isomorphism, the left-most vertical arrow is an isomorphism. From the long exact sequences on $\operatorname{Ext}^i_R(-, M)$ and using the induction hypothesis for $v = j - 1$, we see that $\operatorname{Ext}^i_R(\mathrm{C}_j(Q), M) \cong \operatorname{Ext}^i_R(\mathrm{C}_j(P), M)$ for all $i \geqslant 1$ by the Five Lemma.

Consider now the commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathrm{B}_j(P) & \longrightarrow & P_j & \longrightarrow & \mathrm{C}_j(P) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathrm{B}_j(Q) & \longrightarrow & Q_j & \longrightarrow & \mathrm{C}_j(Q) & \longrightarrow & 0 \end{array}$$

where again the vertical maps are induced by f. From the induced long exact sequences on $\operatorname{Ext}_R^i(-,M)$, the isomorphisms $\operatorname{Ext}_R^i(\mathrm{C}_j(Q),M) \cong \operatorname{Ext}_R^i(\mathrm{C}_j(P),M)$ for all $i \geqslant 1$, and $\operatorname{Ext}_R^i(P_j,M) = \operatorname{Ext}_R^i(Q_j,M) = 0$ for all $i \geqslant 1$ by [9, Lemma 2.1], we obtain that $\operatorname{Ext}_R^i(\mathrm{B}_j(Q),M) \cong \operatorname{Ext}_R^i(\mathrm{B}_j(P),M)$ for all $i \geqslant 1$. □

Theorem 3.3. *Let M be a nonzero object in $\mathsf{D}_+(R)$. Then*

$$\operatorname{level}_R^{\mathsf{G}} M \geqslant \operatorname{Gpd}_R M - \sup M + 1.$$

Moreover, if R is Noetherian and M is in $\mathsf{D}_+^f(R)$, then

$$\operatorname{level}_R^{\widetilde{\mathsf{G}}} M \geqslant \operatorname{Gpd}_R M - \sup M + 1.$$

Proof. We prove the first statement. The second statement is proved similarly.

We may assume $\operatorname{level}_R^{\mathsf{G}}(M) < \infty$. Hence $\operatorname{Gpd}_R M < \infty$ by Proposition **??**. Set $n := \sup M$, $\ell := \inf M$ and $g := \operatorname{Gpd}_R M$. Certainly $g \geqslant n \geqslant \ell$ as $M \not\simeq 0$. If $g = n$ then the inequality is clear, as $M \not\simeq 0$. Suppose now that $g > n$. By [10, Theorem 3.1], there exists an R-complex X such that $X \simeq M$ in $\mathsf{D}_+(R)$, $X_i = 0$ for $i > g$ or $i < \ell$, X_i is projective for all $i \neq n$, and X_n is Gorenstein projective. Without loss of generality, we may replace M with X in the theorem.

For any integer i let $\phi_i : X_{\geqslant i} \to X_{\geqslant i+1}$ be the natural map of truncated complexes.

Claim 1: For all $i \geqslant n$ we have ϕ_i is G-ghost.

Proof of Claim 1: It suffices to prove that for all $i \geqslant n$, the maps $\operatorname{Ext}_R^j(A, X_{\geqslant i}) \to \operatorname{Ext}_R^j(A, X_{\geqslant i+1})$ are zero for all j and all $A \in \mathsf{G}$. Note that $X_{\geqslant i} \simeq \Sigma^i\, \mathrm{C}_i(X)$ for all $i \geqslant n$. Hence, it suffices to prove that for any $i \geqslant n$ the map $\operatorname{Ext}_R^j(A, \mathrm{C}_i(X)) \to \operatorname{Ext}_R^{j+1}(A, \mathrm{C}_{i+1}(X))$ is zero for all j and all $A \in \mathsf{G}$. Since A and $\mathrm{C}_i(X)$ are modules, this is clear for all $j < 0$. If $j \geqslant 0$ and $i \geqslant n$, we note that $\operatorname{pd}_R \mathrm{C}_{i+1}(X) < \infty$ since X_k is projective for all $k \geqslant i+1$. Hence, $\operatorname{Ext}_R^{j+1}(A, \mathrm{C}_{i+1}(X)) = 0$ for all $A \in \mathsf{G}$ by [9, Lemma 2.1].

Let ρ be the natural truncation map $X \to X_{\geqslant n}$ and $\phi'_n = \phi_n\rho$. As ϕ_n is G-ghost so is ϕ'_n. Now let $\psi = \phi_{g-1}\phi_{g-2}\cdots\phi_{n+1}\phi'_n : X \to X_{\geqslant g}$. Then ψ is a composition of $g-n$ G-ghost maps.

Claim 2: ψ induces a nonzero morphism in $\mathsf{D}_+(R)$.

Proof of Claim 2: Suppose $\psi = 0$ in $\mathsf{D}_+(R)$. Choose a semi-projective resolution $\sigma : P \to X$ with $\inf P^{\#} = \inf X^{\#} = \ell$ (cf. [2, 1.7]). Then $\psi\sigma : P \to X_{\geqslant g}$ induces the zero morphism $\mathsf{D}_+(R)$. Since P is semi-projective, this implies $\psi\sigma$ is null-homotopic. This means there exists a map $\tau : P_{g-1} \to X_g$ such that $\sigma_g = \tau\partial_g^P$ where $\partial_g^P : P_g \to P_{g-1}$ is the nth differential of the complex P. (Here we are using that $X_i = 0$ for $i > g$.) Hence we obtain the following diagram where the squares commute and $\tau\overline{\partial_g^P} = \overline{\sigma}_g$:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathrm{C}_g(P) & \xrightarrow{\overline{\partial^P_g}} & P_{g-1} & \longrightarrow & \mathrm{C}_{g-1}(P) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \overline{\sigma}_g} & \tau \swarrow & \downarrow{\scriptstyle \sigma_{g-1}} & & \downarrow{\scriptstyle \overline{\sigma}_{g-1}} & & \\
0 & \longrightarrow & X_g & \xrightarrow[\partial^X_g]{} & X_{g-1} & \longrightarrow & \mathrm{C}_{g-1}(X) & \longrightarrow & 0
\end{array}$$

Note that both rows are exact, as $g > n = \sup X = \sup P$. Now $\sigma : P \to X$ is a quasi-isomorphism of complexes of Gorenstein projective modules and where $\inf P = \inf X$ is finite. As X_g is projective we have by Lemma 3.2(a) that the induced map $\operatorname{Ext}^1_R(\mathrm{C}_{g-1}(X), X_g) \to \operatorname{Ext}^1_R(\mathrm{C}_{g-1}(P), X_g)$ is an isomorphism. Hence, by Lemma 3.1 we get that the map ∂^X_g splits. Thus, $\mathrm{C}_{g-1}(X)$ is isomorphic to a direct summand of X_{g-1}, which is Gorenstein projective. Hence, $\mathrm{C}_{g-1}(X)$ is Gorenstein projective. As $g-1 \geqslant n = \sup X$, this implies $g = \operatorname{Gpd}_R X \leqslant g-1$ by Proposition 2.3, a contradiction.

Since $\psi : X \to X_{\geqslant g}$ is a composition of $g-n$ G-ghost maps and is nonzero in $\mathsf{D}_+(R)$, we have by the Ghost Lemma (Theorem 2.11) that $\operatorname{level}^{\mathsf{G}}_R(X) \geqslant g-n+1$. As $\operatorname{level}^{\mathsf{G}}_R X = \operatorname{level}^{\mathsf{G}}_R M$, this concludes the proof. □

Remark 3.4. After a preliminary version of this paper appeared, R. Takahashi pointed out to the authors that in the case M is a module, an alternative proof of Theorem 3.3 can be obtained using [3, Theorem 1.2].

As an immediate consequence, we have the following generalization of [7, Proposition 4.5] and [1, Cor 2.2]:

Corollary 3.5. *For a nonzero R-module M we have*

$$\operatorname{level}^{\mathsf{G}}_R M = \operatorname{Gpd}_R M + 1.$$

If in addition R is Noetherian and M is finitely generated, we have

$$\operatorname{level}^{\widetilde{\mathsf{G}}}_R M = \operatorname{Gpd}_R M + 1.$$

Proof. These statements follow readily from Theorem 3.3 and Corollary 2.8. □

An interesting question is whether $\operatorname{level}^{\mathsf{P}}_R M$ (respectively, $\operatorname{level}^{\mathsf{G}}_R M$) coincides with $\operatorname{level}^{\widetilde{\mathsf{P}}}_R M$ (respectively, $\operatorname{level}^{\widetilde{\mathsf{G}}}_R M$) for complexes M in $\mathsf{D}^f_+(R)$. We get an affirmative answer for P-level using the converse Coghost Lemma:

Proposition 3.6. *Let R be Noetherian and M an object in $\mathsf{D}^f_+(R)$. Then*

$$\operatorname{level}^{\mathsf{P}}_R M = \operatorname{level}^{\widetilde{\mathsf{P}}}_R M.$$

Proof. We first note that if M is not (homologically) bounded above both quantities must be infinite. Thus, we may assume M is in $\mathsf{D}^f_b(R)$. The inequality $\operatorname{level}^{\mathsf{P}}_R M \leqslant \operatorname{level}^{\widetilde{\mathsf{P}}}_R M$ is clear. The reverse inequality is clear if $\operatorname{level}^{\widetilde{\mathsf{P}}}_R M \leqslant 1$. Suppose $\operatorname{level}^{\widetilde{\mathsf{P}}}_R M = n \geqslant 2$. It suffices to prove $\operatorname{level}^{\mathsf{P}}_R M \geqslant n$. By the converse coghost lemma, there exist $\widetilde{\mathsf{P}}$-coghost maps $f_i : M_i \to M_{i+1}$ for $0 \leqslant i \leqslant n-1$ in $D^f_b(R)$ with $M_n = M$ and $f_{n-1}f_{n-2}\cdots f_0$ a nonzero morphism. As noted in Remark 2.10, the maps f_i are also P-coghost. Since $f_{n-1}f_{n-2}\cdots f_0$ is a nonzero morphism we obtain that $\operatorname{level}^{\mathsf{P}}_R M \geqslant n$ by the Coghost Lemma. □

For the case of G-level, we can answer the question affirmatively in the case of modules using Corollary 3.5:

Proposition 3.7. *Let M be an R-module.*
(a) If $\operatorname{level}_R^{\mathsf{P}} M < \infty$ *then* $\operatorname{level}_R^{\mathsf{P}} M = \operatorname{level}_R^{\mathsf{G}} M$.
(b) If R is Noetherian and M is finitely generated then $\operatorname{level}_R^{\mathsf{G}} M = \operatorname{level}_R^{\widetilde{\mathsf{G}}} M$.
(c) If R is Noetherian and $\operatorname{level}_R^{\widetilde{\mathsf{P}}} M < \infty$*, then* $\operatorname{level}_R^{\widetilde{\mathsf{P}}} M = \operatorname{level}_R^{\widetilde{\mathsf{G}}} M$.

Proof. We may assume M is nonzero. For the first assertion, we have by [1, Cor 2.2] that $\operatorname{level}_R^{\mathsf{P}} M = \operatorname{pd}_R M + 1$. Thus, $\operatorname{pd}_R M < \infty$. Then by [11, Proposition 2.27], we have $\operatorname{Gpd}_R M = \operatorname{pd}_R M$. The result now follows from Corollary 3.5. The second assertion follows immediately from Corollary 3.5. The third statement follows from the first two, along with Proposition 3.6. □

Another consequence of Theorem 3.3 is the following characterization of Gorenstein local rings:

Corollary 3.8. *Let R be a local Noetherian ring with residue field k. The following conditions are equivalent:*
(a) R is Gorenstein;
(b) $\operatorname{level}_R^{\mathsf{G}} k < \infty$*;*
(c) $\operatorname{level}_R^{\mathsf{G}} k = \dim R + 1$*;*
(d) $\operatorname{level}_R^{\mathsf{G}} M \leqslant \dim R + \sup M - \inf M + 1$ *for all nonzero complexes M in* $\mathsf{D}_+(R)$.
(e) $\operatorname{level}_R^{\widetilde{\mathsf{G}}} M \leqslant \dim R + \sup M - \inf M + 1$ *for all nonzero complexes M in* $\mathsf{D}_+^f(R)$.

Proof. This follows from Theorem 2.4, Theorem 3.3, Corollary 2.8 and Corollary 3.5. □

In [4], the following upper bound on level with respect to R is proved:

Theorem 3.9. ([4, Theorem 5.5]) *Let R be Noetherian and M a nonzero complex in* $\mathsf{D}_b^f(R)$*. Then*

$$\operatorname{level}_R^{\widetilde{\mathsf{P}}} M \leqslant \operatorname{pd}_R \mathrm{H}(M) + 1,$$

where $\mathrm{H}(M)$ *is considered as a module concentrated in degree zero, not as a complex. In particular, if R is regular of finite dimension, then* $\operatorname{level}_R^R(M) \leqslant \dim R + 1$.

One can ask whether either inequality holds if projective dimension is replaced by Gorenstein projective dimension, and level with respect to $\widetilde{\mathsf{P}}$ replaced by level with respect to $\widetilde{\mathsf{G}}$. The answer is no, as the following example demonstrates:

Example 3.10. Let k be a field and $R = k[x]/(x^2)$ and F the complex

$$0 \to R \xrightarrow{x} R \to 0,$$

where the modules R sit in homological degrees 1 and 0. Note that $\mathrm{H}(F)$ is finitely generated and nonzero. Since R is a zero-dimensional Gorenstein ring, we have $\operatorname{Gpd}_R \mathrm{H}(F) = 0$. (As in Theorem 3.9, we are considering $\mathrm{H}(F)$ as a module concentrated in degree zero, not as a complex.) We claim that $\operatorname{level}_R^{\widetilde{\mathsf{G}}}(F) = 2$. From Corollary 2.8, we have that $\operatorname{level}_R^{\widetilde{\mathsf{G}}}(F) \leqslant \operatorname{Gpd}_R F + 1 \leqslant 2$. Suppose $\operatorname{level}_R^{\widetilde{\mathsf{G}}}(F) \leqslant 1$. Then $F \simeq T$ in $\mathsf{D}(R)$, where T is a bounded complex of finitely generated Gorenstein projective modules with zero differentials. Since T has zero differentials, we have $T \simeq \mathrm{H}(T)$ in $\mathsf{D}(R)$. Thus, $F \simeq T \simeq \mathrm{H}(T) \simeq \mathrm{H}(F)$ in $\mathsf{D}(R)$. Since F is semi-projective, this means there exists a quasi-isomorphism $\sigma : F \to \mathrm{H}(F)$. Let $t = \sigma_1(1) \in \mathrm{H}(F)_1 = xR$. The induced map on homology $\sigma_1^* : xR \to xR$ is multiplication by t, which is the zero map. This contradicts that σ_1^* is an isomorphism. Hence, $\mathrm{level}_R^{\widetilde{\mathsf{G}}}(F) = 2$.

The following result provides a global bound on the levels of complexes with respect to G over a Gorenstein ring:

Theorem 3.11. *Let R be a Noetherian Gorenstein ring and M a complex in $\mathsf{D}_b(R)$. Then*

$$\mathrm{level}_R^{\mathsf{G}} M \leqslant 2(\dim R + 1).$$

Similarly, if M is a complex in $\mathsf{D}_b^f(R)$ then $\mathrm{level}_R^{\widetilde{\mathsf{G}}} M \leqslant 2(\dim R + 1)$.

Proof. Let Z and B be the subcomplexes of M consisting of the cycles and boundaries of M, respectively. The we have a short exact sequence of complexes

$$0 \to Z \to M \to \Sigma B \to 0,$$

which induces an exact triangle $Z \to M \to \Sigma B \to \Sigma Z$ in $\mathsf{D}_b(R)$. Note that Z and B are bounded complexes with zero differentials. Furthermore, if M is in $\mathsf{D}_b^f(R)$, we can assume Z and B are finitely generated by replacing M, if necessary, with a semi-projective resolution consisting of finitely generated projective modules in each degree. As G is Gorenstein, $\mathrm{Gpd}_R L \leqslant \dim R$ for every R-module L. By Corollary 2.8, $\mathrm{level}_R^{\mathsf{G}} L \leqslant \dim R + 1$, and if L is finitely generated, $\mathrm{level}_R^{\widetilde{\mathsf{G}}} L \leqslant \dim R + 1$. Since level is invariant under direct sums and suspensions, we see that $\mathrm{level}_R^{\mathsf{G}} Z$ and $\mathrm{level}_R^{\mathsf{G}} B$ are each bounded above by $\dim R + 1$; similarly for $\mathrm{level}_R^{\widetilde{\mathsf{G}}} Z$ and $\mathrm{level}_R^{\widetilde{\mathsf{G}}} B$ in the case M is in $\mathsf{D}_b^f(R)$. The theorem now follows by part (5) of Proposition 2.5. □

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