Math 915: Midterm

Due: Friday, October 31, 5pm

Instructions: Consider this as another homework assignment, except that you are to work alone. Everything you need to solve the problems below can be found in our class notes or in previous homework problems. You may also consult Eloísa Grifo's 915 course notes from 2023. However, please don't go combing the internet for solutions, consult AI chatbots, or the like. If you are stuck on a problem (or several), I am more than happy to provide hints in person or through email.

I will post the .tex code for these problems for you to use if you wish to type your homework. If you prefer not to type, please *write neatly.* As a matter of good proof writing style, please use complete sentences and correct grammar.

Throughout, R denotes a commutative ring with identity.

Problem 1. Consider the $\mathbb{Z}$-module $M := \mathbb{Q}/\mathbb{Z}_{(2)}$. For $n \geq 0$ let $A_n := \mathbb{Z}\overline{\frac{1}{2^n}}$, where $\overline{u} = u + \mathbb{Z}$ for any $u \in \mathbb{Q}$.

(a) Prove that $A_n \subsetneq A_{n+1}$ for all $n \geq 0$.

(b) Prove that every proper $\mathbb{Z}$-submodule of M is equal to A_n for some n.

(c) Conclude that M is Artinian as a $\mathbb{Z}$-module but not Noetherian.

Problem 2. Let (R, m, k) be a local ring and M a finitely generated R-module.

(a) Prove that $\operatorname{pd}_R M = \sup\{i \mid \operatorname{Ext}^i_R(M, k) \neq 0\}$.

(b) Prove that if $\operatorname{id}_R k < \infty$ then $\operatorname{pd}_R M < \infty$ for all f.g R-modules M.

Problem 3. Let $S = \mathbb{Q}[x, y, z]$ and $T = \mathbb{Z}_{(2)}[x]$.

(a) Prove that $\{x, y - xy, z - xz\}$ is an S-sequence but $\{y - xy, z - xz, x\}$ is not an S-sequence.

(b) Prove that $\{2, x\}$ and $\{2x - 1\}$ are both maximal T-sequences. (Hint: You may use without proof that for any $a \in R$ one has $R[x]/(ax - 1) \cong R_W$, where $W = \{a^n \mid n \geq 0\}$.)

Problem 4. Let M and N be R-modules and $x \in R$ such that x is both R-regular and M-regular (i.e., $\{x\}$ is both an R-sequence and an M-sequence). Assume also that $xN = 0$.

(a) Prove that $\operatorname{Tor}^R_i(M, R/(x)) = 0$ for $i \geq 1$.

(b) Let F be a free resolution of M. Prove that $F/xF \cong F \otimes_R R/(x)$ is a free $R/(x)$-resolution of M/xM. (Hint: Use part (a).)

(c) Prove that $F \otimes_R N \cong F/xF \otimes_{R/(x)} N$. (Hint: $N \cong N/xN \cong R/(x) \otimes_{R/(x)} N$.)

(d) Prove that $\operatorname{Tor}^R_i(M, N) \cong \operatorname{Tor}^{R/(x)}_i(M/xM, N)$ for all i.

Problem 5. Let (R, m, k) be a local ring and M a finitely generated R-module. Let $\underline{x} = x_1, \ldots, x_n \in m$ be both an R-sequence and an M-sequence. Recall that $\beta^R_i(M) = \dim_k \operatorname{Tor}^R_i(M, k)$.

(a) Prove that $\beta^R_i(M) = \beta^{R/(\underline{x})}_i(M/(\underline{x})M)$ for all $i \geq 0$.

(b) Prove that $\operatorname{pd}_R M = \operatorname{pd}_{R/(\underline{x})} M/(\underline{x})M$.